Advanced Lattice Sieving on GPUs, with Tensor Cores

Léo Ducas, Marc Stevens, **Wessel van Woerden** (CWI).

Centrum Wiskunde & Informatica

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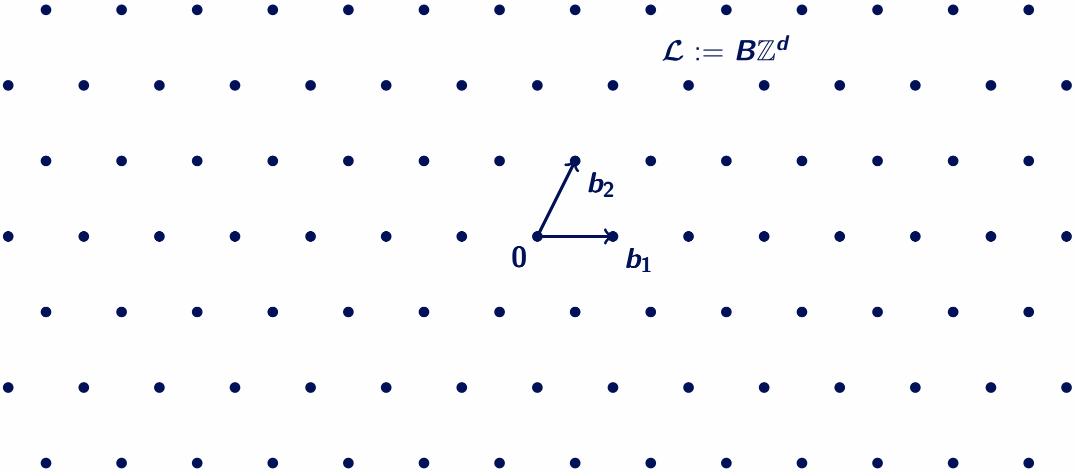
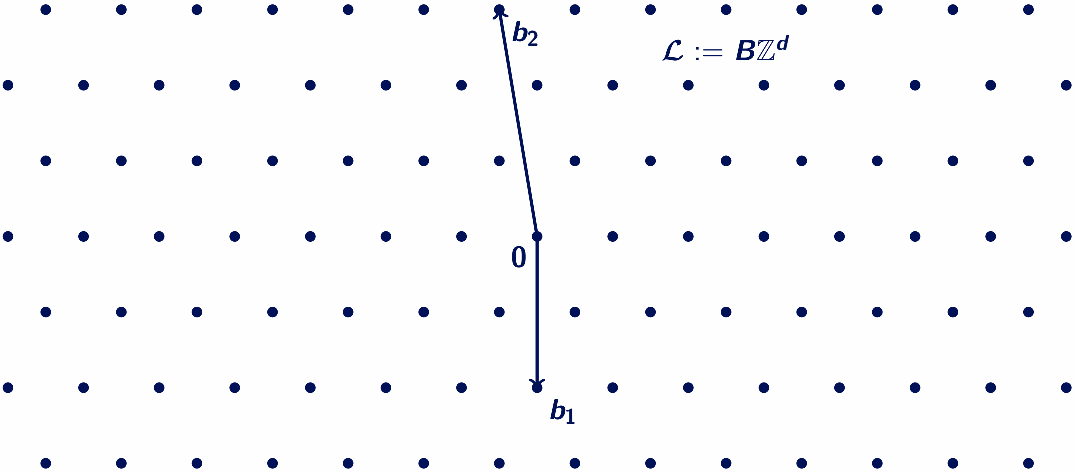
- First optimized implementation of the asymptotic best known sieve [BDGL].

Lattice

$$\mathcal{L} := \boldsymbol{B}\mathbb{Z}^{\boldsymbol{d}}$$





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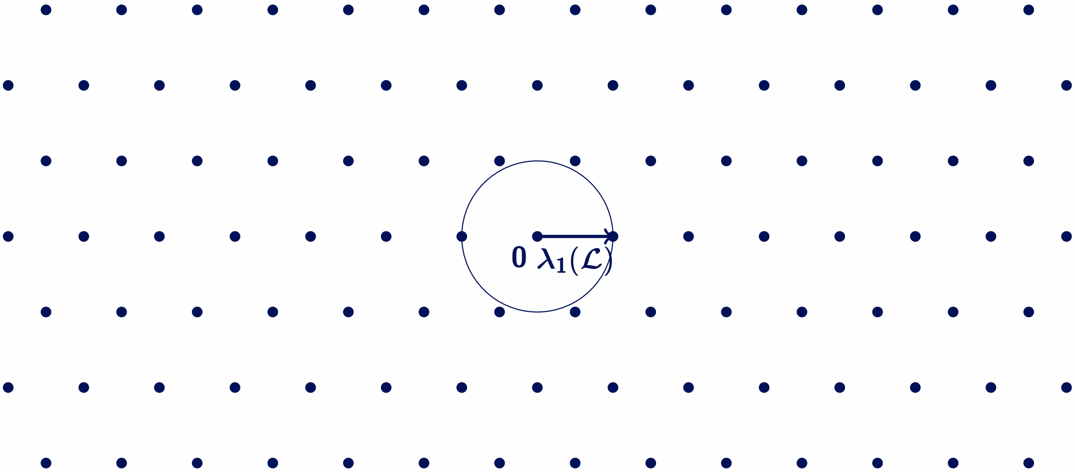
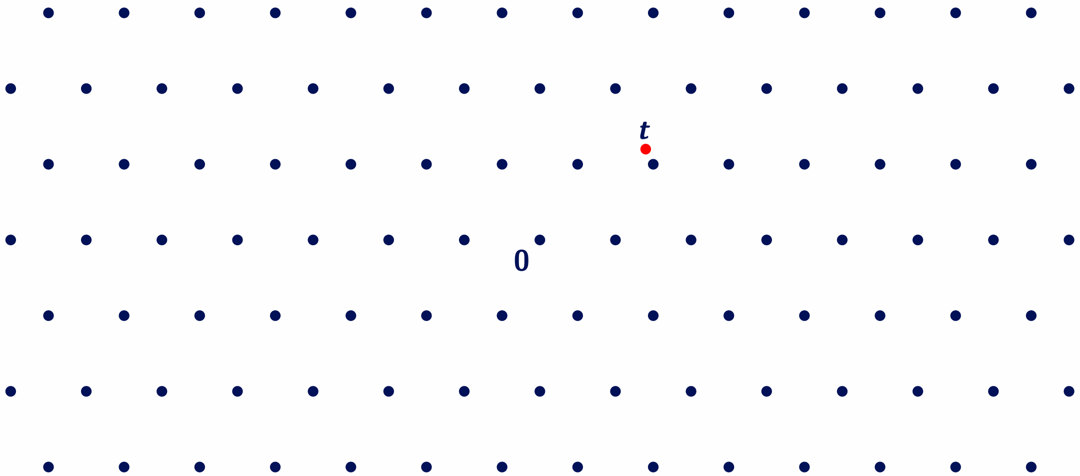
Shortest Vector Problem



Bounded Distance Decoding



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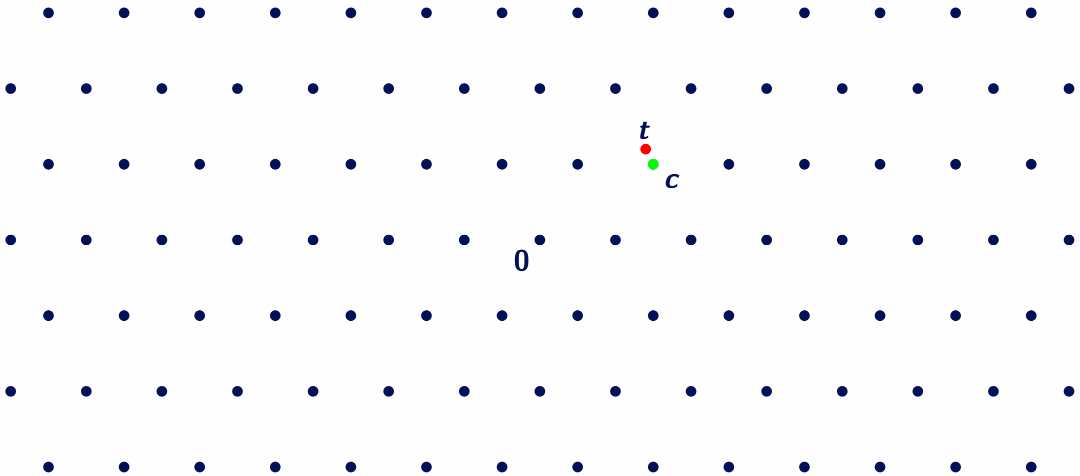

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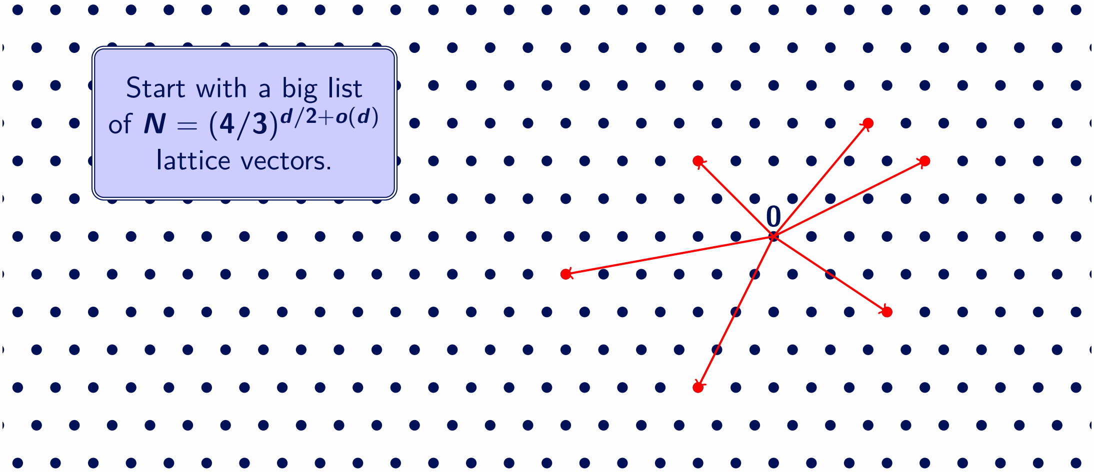
- Given: 'Random' $\boldsymbol{d}$-dimensional lattice $\boldsymbol{\mathcal{L}}$ (Goldstein and Mayer)
- Goal: Find a $\boldsymbol{v} \in \boldsymbol{\mathcal{L}}$ s.t.

$$\|\boldsymbol{v}\| \leq \mathbf{1.05} \cdot \mathrm{GH}(\boldsymbol{\mathcal{L}}) \approx \mathbf{1.05} \cdot \boldsymbol{\lambda_1}(\boldsymbol{\mathcal{L}}).$$



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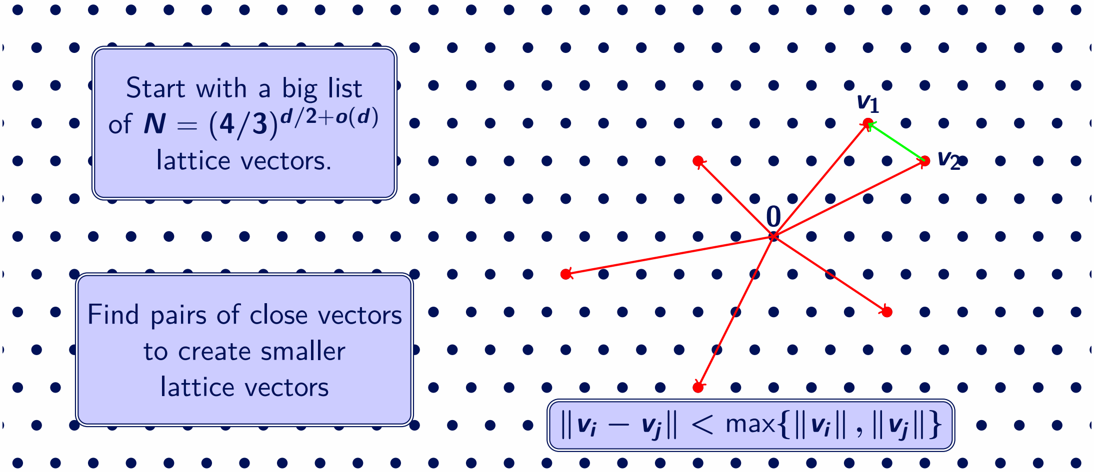
Find pairs of close vectors to create smaller lattice vectors



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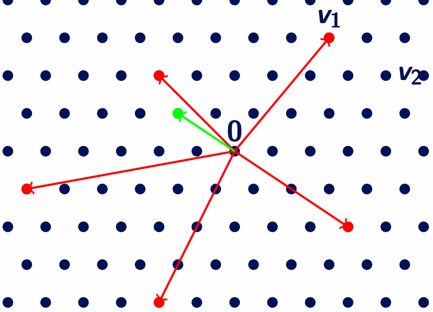

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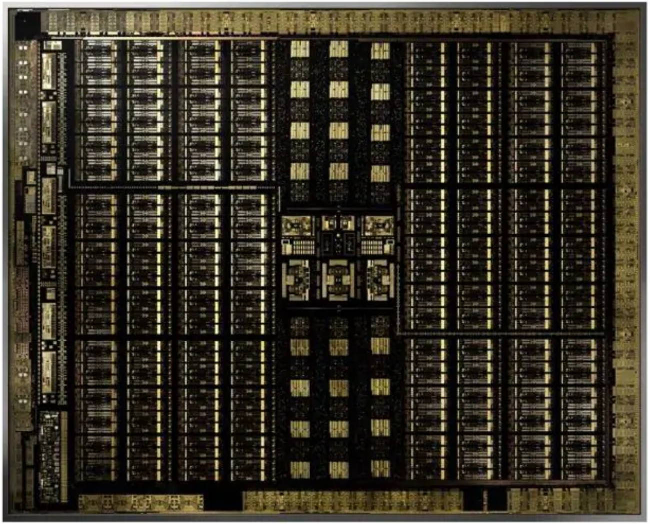
Graphics Processing Unit (GPU)



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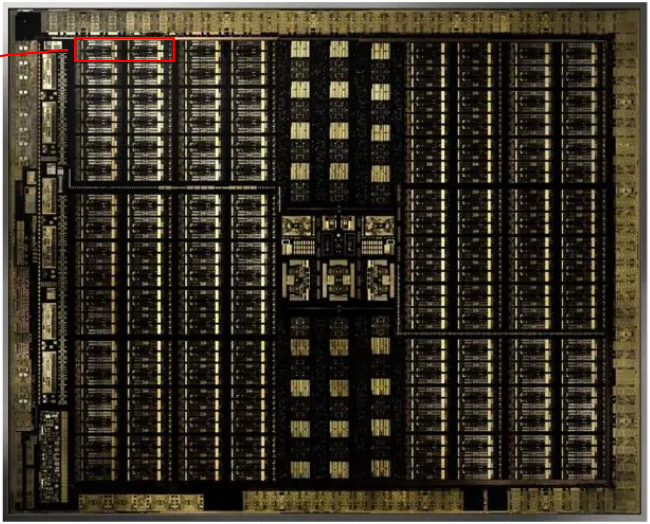
64 FP32 cores
64 INT32 cores
8 Tensor cores.

Thousands of cores.



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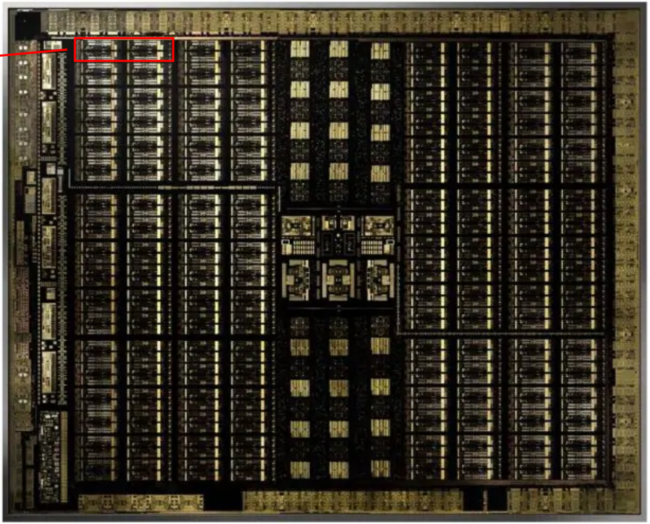
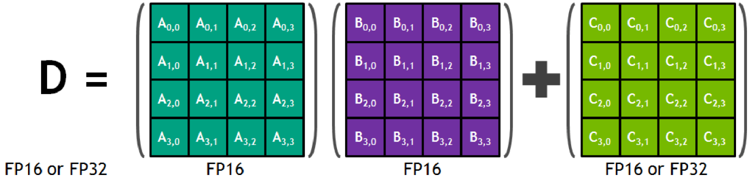
Per **32** cores:
Single Instruction
Multiple Data



Tensor Cores



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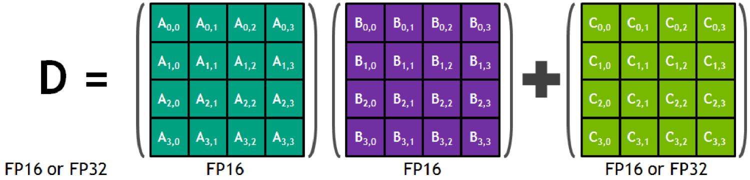

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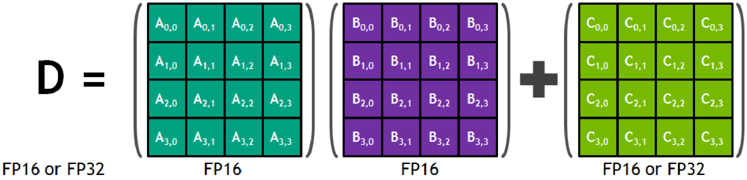

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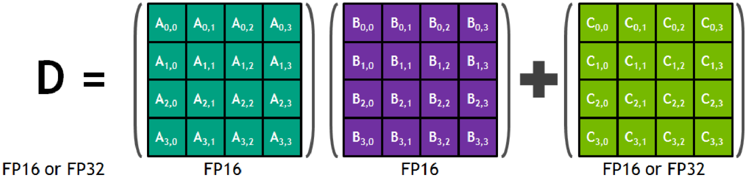

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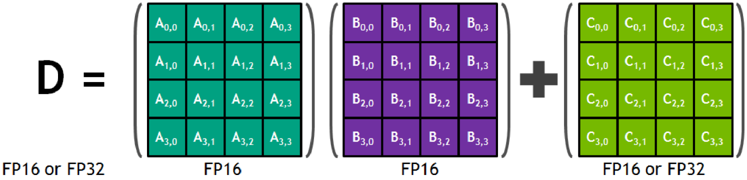

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- Newest model > **300** 16-bit **Tflops**.
- The current best CPU would reach at most ≈ **5** 16-bit **Tflops**.

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G6K, Albrecht et al. 2019

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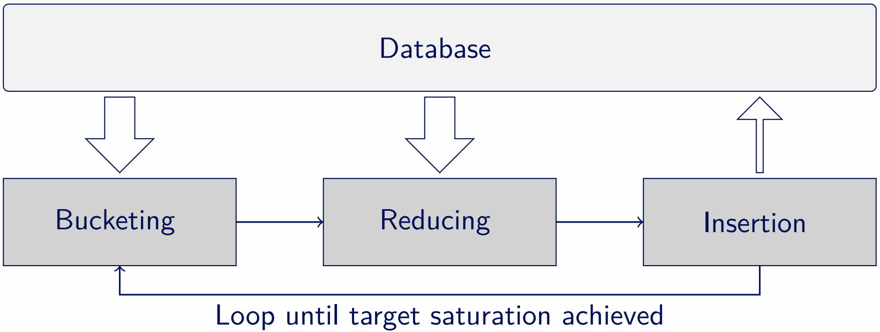
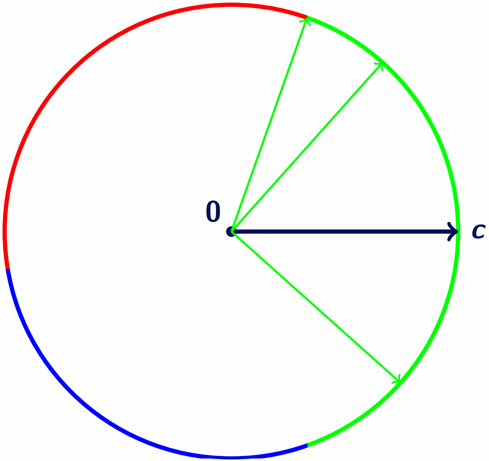
- Enumeration: dimension **152** using **800.000** core hours!.

Advanced Sieving on GPUs

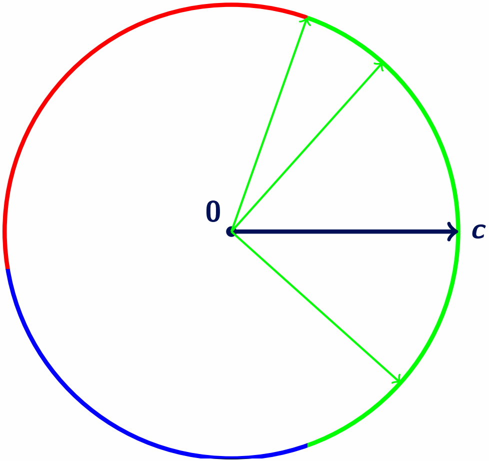
Sieving Process





Partition the sphere.

Bucketing



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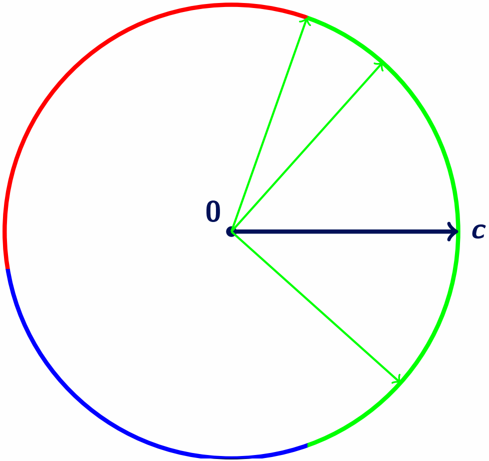

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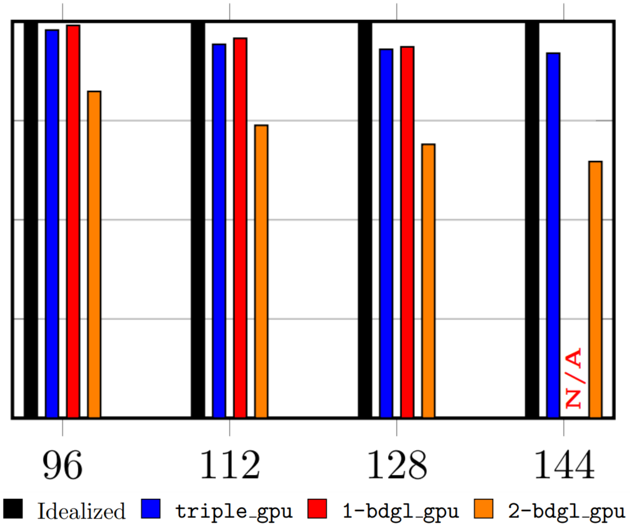
  - AVX2 CPU implementation merged into G6K, fastest CPU sieve.

Bucketing Quality



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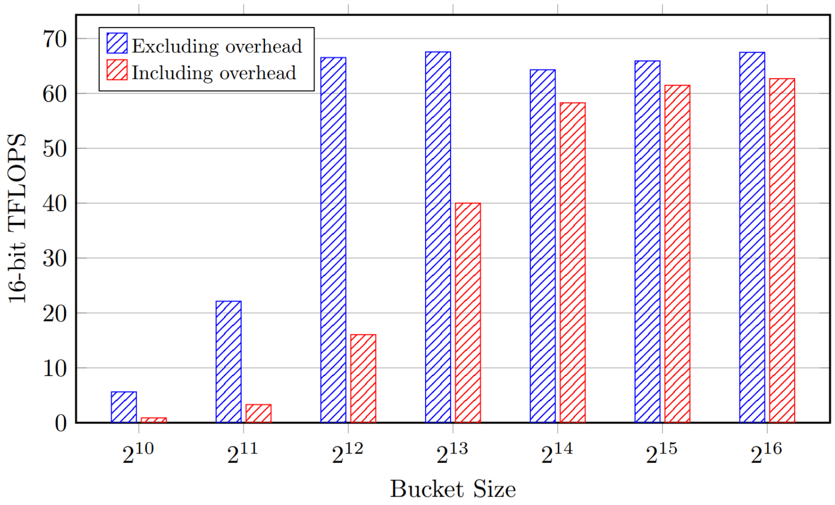
- FLOP: $\boldsymbol{O}(\boldsymbol{d} \cdot \boldsymbol{B}^2)$, data: $\boldsymbol{O}(\boldsymbol{d} \cdot \boldsymbol{B})$, ratio improves for larger bucket size $\boldsymbol{B}$.

Amortizing data throughput



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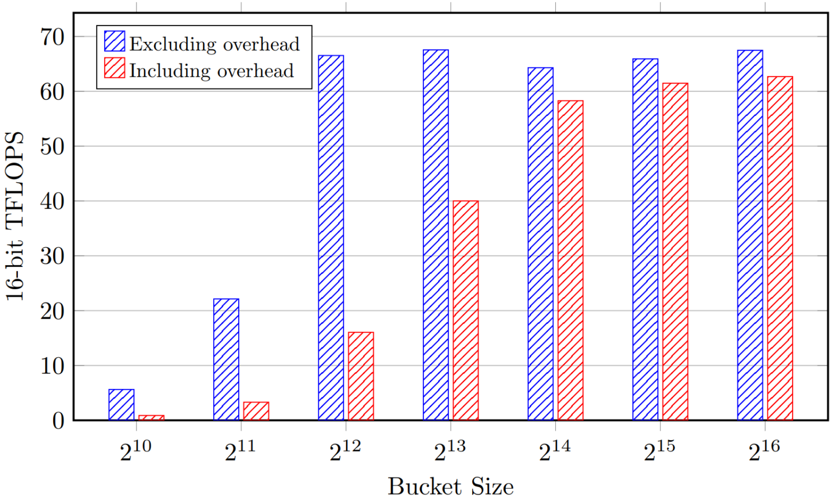

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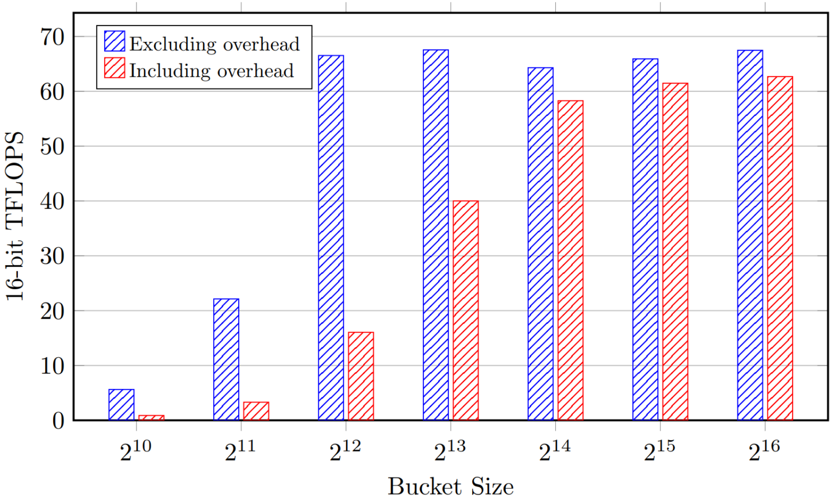

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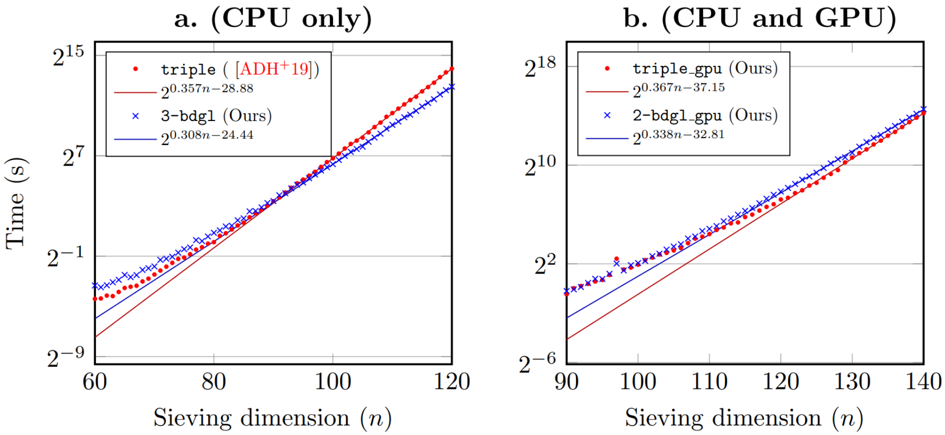
- Large buckets to reach optimal performance.

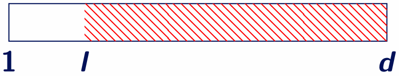
BGJ1 vs BDGL



a. (CPU only)
triple ([ADH+19])
$2^{0.357n-28.88}$
3-bdgl (Ours)
$2^{0.308n-24.44}$
Time (s)
2^{15}
2^{7}
2^{-1}
2^{-9}
60
80
100
120
Sieving dimension (n)
b. (CPU and GPU)
triple_gpu (Ours)
$2^{0.367n-37.15}$
2-bdgl_gpu (Ours)
$2^{0.338n-32.81}$
2^{18}
2^{10}
2^{2}
2^{-6}
90
100
110
120
130
140
Sieving dimension (n)

Dimensions for Free [Duc18]

- Sieve in a projected sublattice $\pi_l(\mathcal{L})$ (projected away from first l basis vectors).



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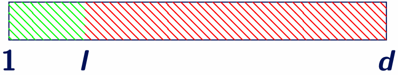

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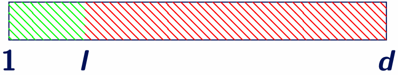

- Finds the shortest vector for $l = O\left(\frac{d}{\log(d)}\right)$.

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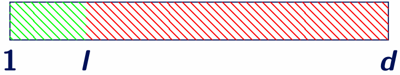

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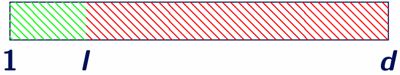

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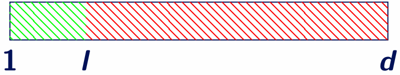

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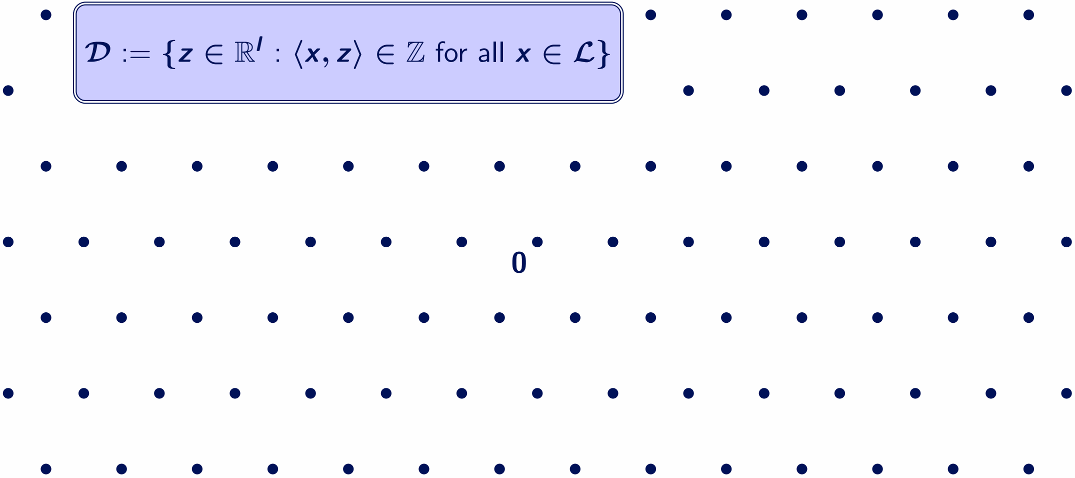
- On the fly lifting: lift any shortish vector we encounter.
- Can we efficiently detect if $v_i - v_j$ might lift to a short vector [BDD problem]?

$$\mathcal{D} := \{\boldsymbol{z} \in \mathbb{R}^{\boldsymbol{l}} : \langle \boldsymbol{x}, \boldsymbol{z} \rangle \in \mathbb{Z} \text{ for all } \boldsymbol{x} \in \mathcal{L}\}$$



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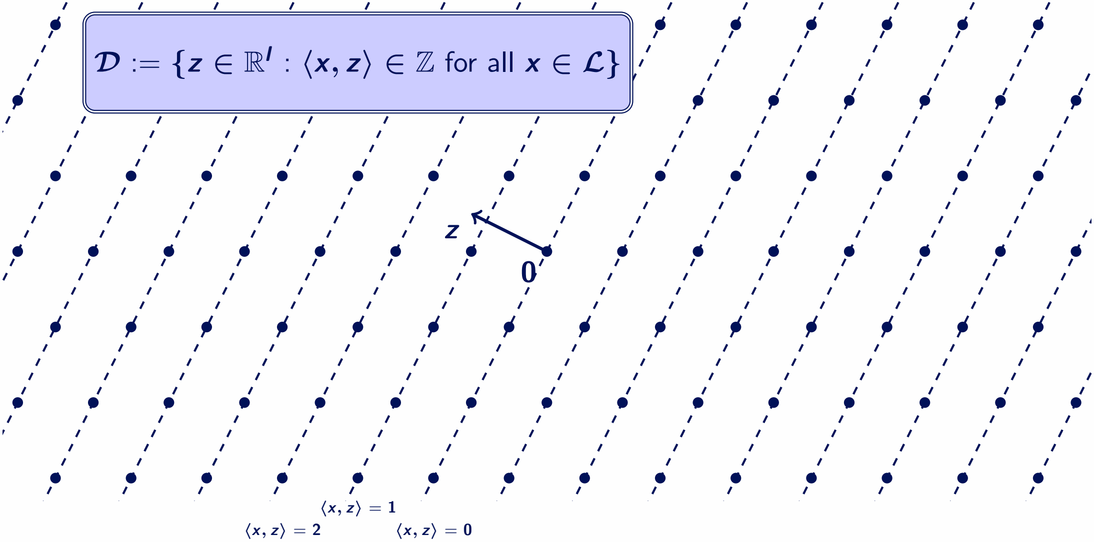

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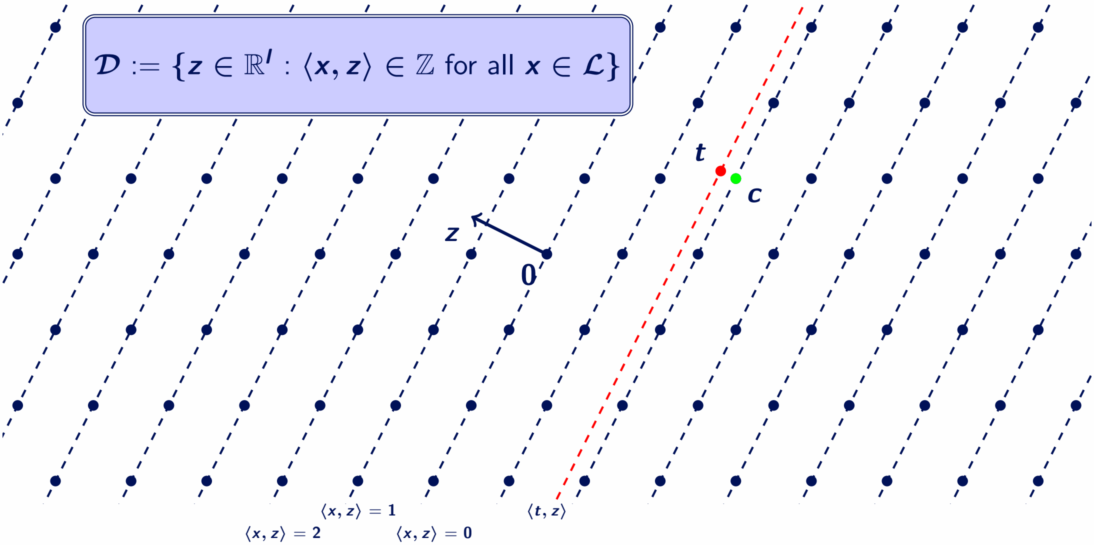

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- $\boldsymbol{H}(\boldsymbol{t_i} - \boldsymbol{t_j}) = \boldsymbol{H}(\boldsymbol{t_i}) - \boldsymbol{H}(\boldsymbol{t_j})$, so $\boldsymbol{O}(\boldsymbol{k})$ operations per pair.
- Suitable for GPUs.

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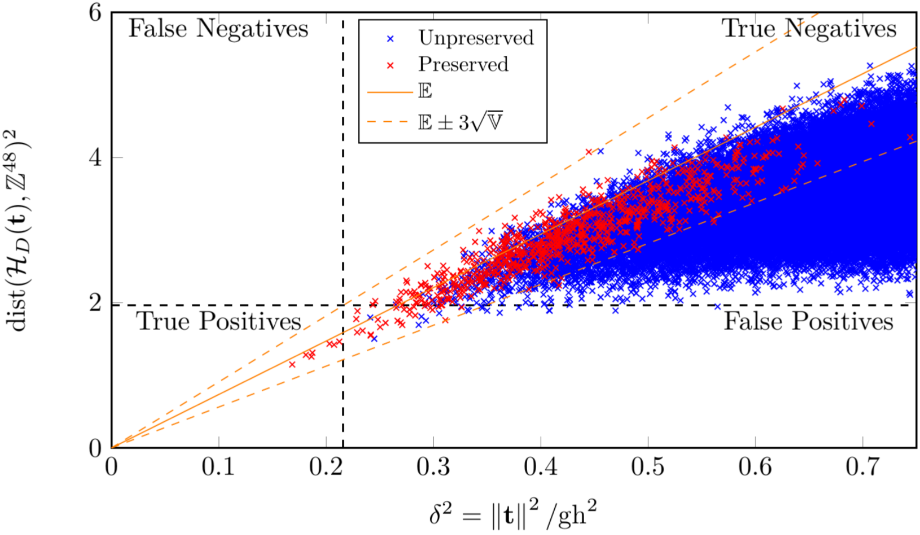

False Negatives
True Negatives
True Positives
False Positives
Unpreserved
Preserved
$\mathbb{E}$
$\mathbb{E} \pm 3\sqrt{\mathbb{V}}$
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
$\mathrm{dist}(\mathcal{H}_D(\mathbf{t}), \mathbb{Z}^{48})^2$
$\delta^2 = \|\mathbf{t}\|^2 / \mathrm{gh}^2$

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  - $[\langle \boldsymbol{y}, \tilde{\boldsymbol{b}}_1 \rangle / \left\| \tilde{\boldsymbol{b}}_1 \right\|, \ldots, \langle \boldsymbol{y}, \tilde{\boldsymbol{b}}_{\boldsymbol{d}} \rangle / \left\| \tilde{\boldsymbol{b}}_{\boldsymbol{d}} \right\|]$. $(4\boldsymbol{d})$

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  - Lift target $\boldsymbol{t}$. (**4l**)
  - Dual Hash (**4l**)
  - Popcount (**d/4**)

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  - Unique Identifier $(\mathbf{8})$
  - Lift target $\boldsymbol{t}$. $(\mathbf{4}\boldsymbol{l})$
  - Dual Hash $(\mathbf{4}\boldsymbol{l})$
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- Remove all except $\boldsymbol{x}$, $\|\boldsymbol{y}\|^2$ and unique identifier.

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  - $\|\boldsymbol{y}\|^2$. (8)
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  - Unique Identifier $(\mathbf{8})$
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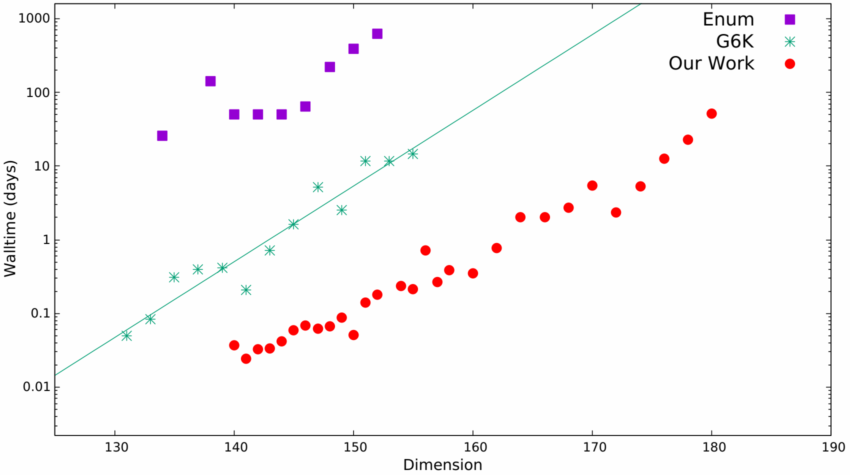
  - Dual Hash $(\mathbf{4}l)$
  - Popcount $(d/\mathbf{4})$
- Remove all except x, $\|\boldsymbol{y}\|^2$ and unique identifier.
- Reduces memory by $\pm\mathbf{60}\%$.
- Compute everything on the GPU, overhead of $O(B \cdot d^2)$ for a bucket size B.

New SVP records

Dimension 180!



maximum RAM size of 1.5TB reached for **180**.

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- Extra benefit of saving memory with negligible overhead.
- BDGL beats BGJ1 in practice on CPUs, but the cross-over for GPUs lies much higher.

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