# On the Lattice Isomorphism Problem, Quadratic Forms, Remarkable Lattices, and Cryptography

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# Motivation

- LWE, SIS, NTRU lattices: versatile, but poor decoding.

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- General identification, encryption and signature scheme based on the Lattice Isomorphism Problem.

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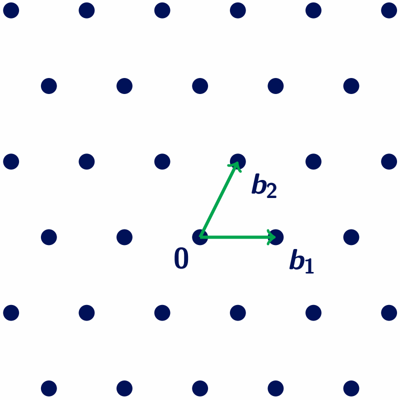
## Contributions

- General identification, encryption and signature scheme based on the Lattice Isomorphism Problem.
- Better lattice $\implies$ better efficiency and security.

Lattice $\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$



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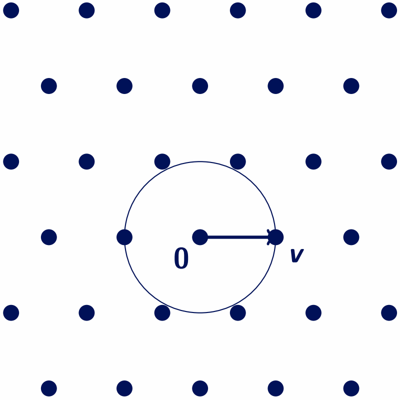


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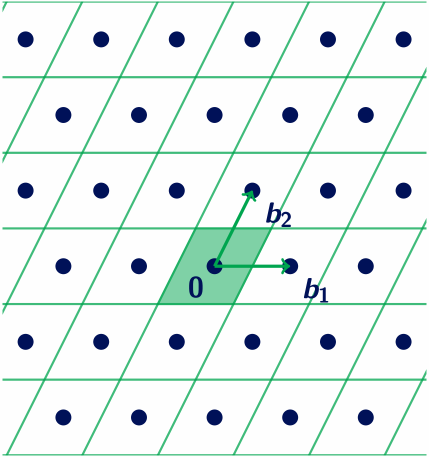


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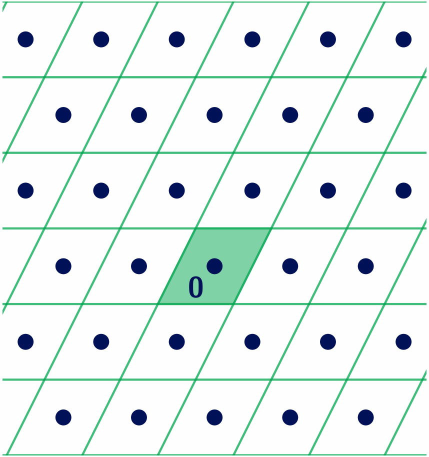


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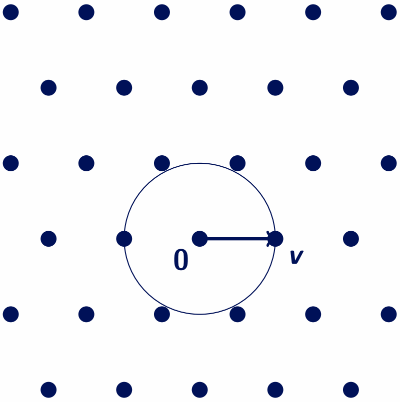
Minkowski's Theorem

$$\lambda_1(\mathcal{L}) \leq \underbrace{2\frac{\det(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}^n)^{1/n}}}_{\mathrm{Mk}(\mathcal{L})} \leq \sqrt{n}\det(\mathcal{L})^{1/n}$$

Lattice $\mathcal{L} \subset \mathbb{R}^n$




**SVP**

Find a *shortest* nonzero vector $\mathbf{v} \in \mathcal{L}$ of length $\lambda_1(\mathcal{L}) \leq \mathsf{Mk}(\mathcal{L})$ .

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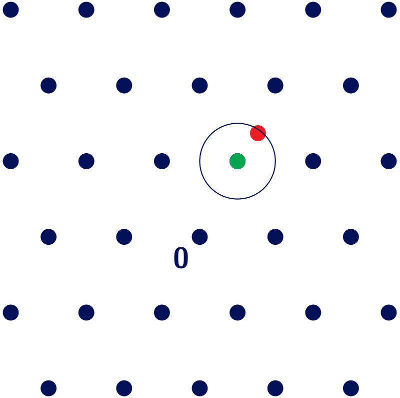


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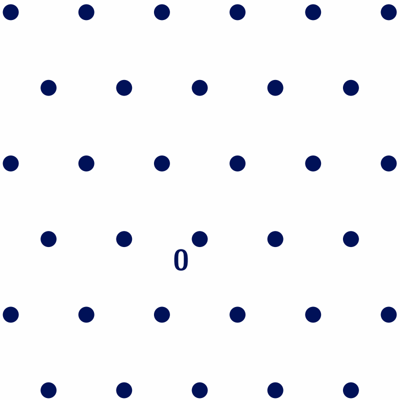
BDD

Given a target $\boldsymbol{t} = \boldsymbol{v} + \boldsymbol{e} \in \mathbb{R}^n$ with $\boldsymbol{v} \in \mathcal{L}$ and $\|\boldsymbol{e}\| < \rho \leq \frac{1}{2}\lambda_1(\mathcal{L}) \leq \frac{1}{2}\mathsf{Mk}(\mathcal{L})$ ,

recover $\boldsymbol{v} \in \mathcal{L}$.

# Hard Problems

Lattice $\mathcal{L} \subset \mathbb{R}^n$

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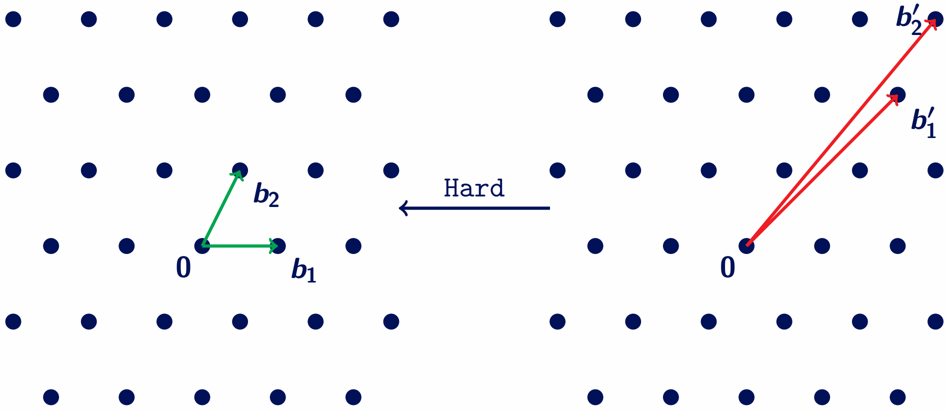
Hardness depends on the *gaps*!

# Encryption, legacy approach

Good basis (Secret key)

Bad basis (Public key)




$B$

$B' = B \cdot U, \; U \in \mathrm{GL}_d(\mathbb{Z})$

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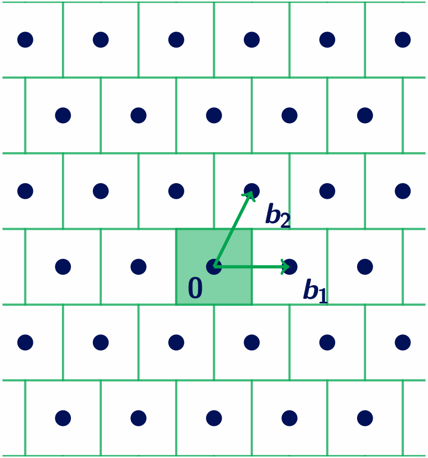


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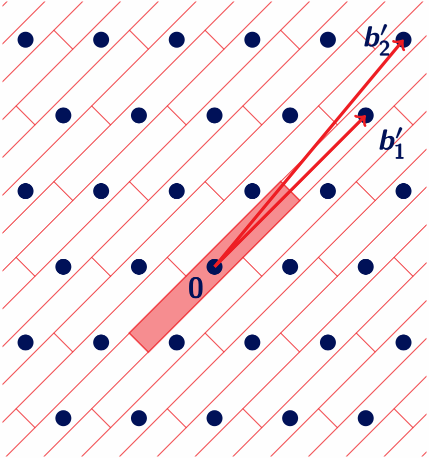


Babai's nearest plane algorithm

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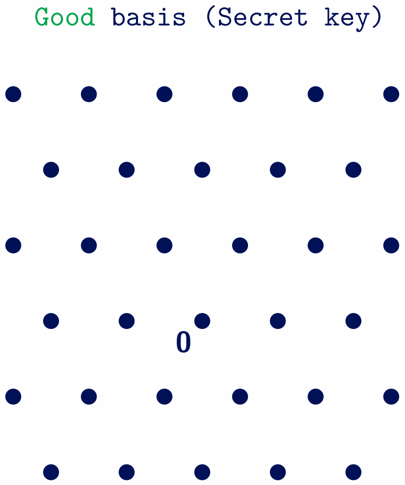


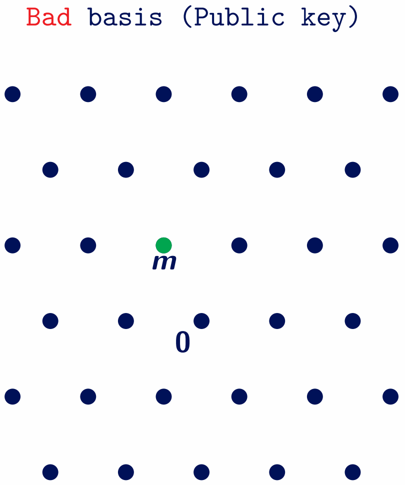

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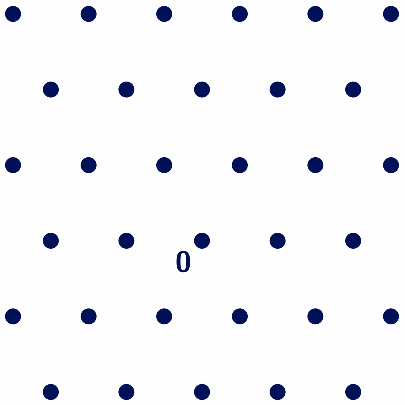


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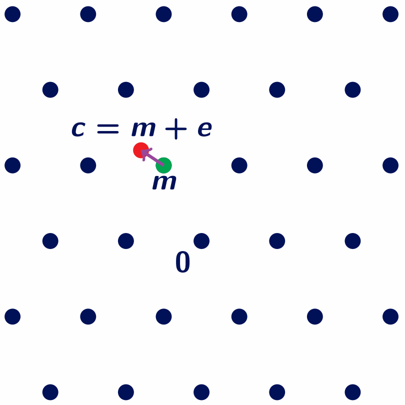


Encrypt by adding a small error

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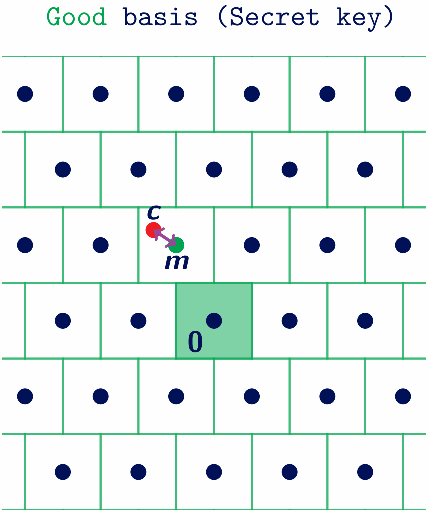


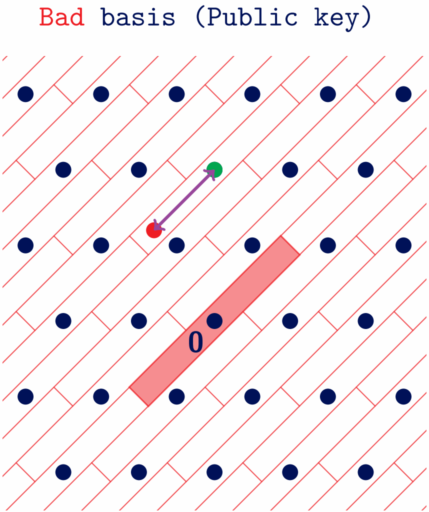


Decrypt using the good basis

## Large gap

Current lattice based crypto relies on hardness of decoding with

$$\mathrm{gap}(\mathcal{L}, \rho) \geq \Omega(\sqrt{n}).$$

Broken by SVP in dimension $\beta \leq n/2 + o(n)$, e.g.
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## An example: Prime Lattice [CR88]

Let $p_1, \ldots, p_n$ be distinct small primes not dividing $m$, we define:

$$\mathcal{L}_{\mathrm{prime}} := \{x = (x_1, \ldots, x_n) \in \mathbb{Z}^n : \prod_i p_i^{x_i} = 1 \bmod m\}.$$

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- With the right parameters $\mathrm{gap}(\mathcal{L}_{\mathtt{prime}}, \rho) = \Theta(\log(n))$ [DP19].

# How to hide the remarkable lattice?

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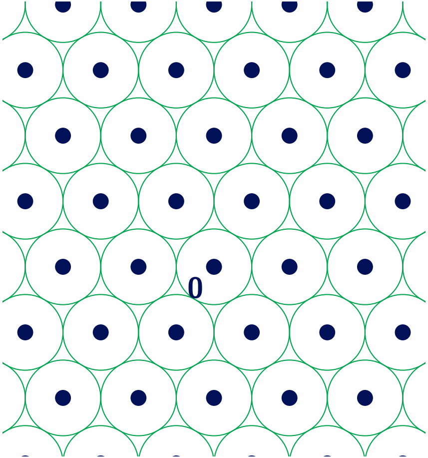


$\mathcal{L}$

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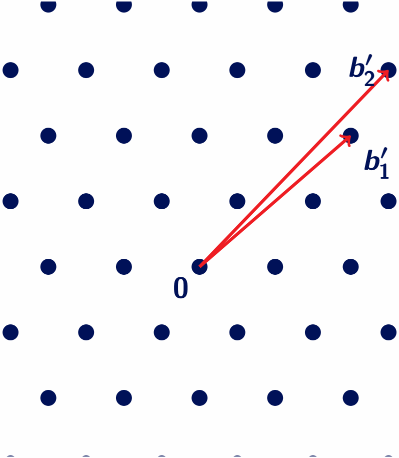


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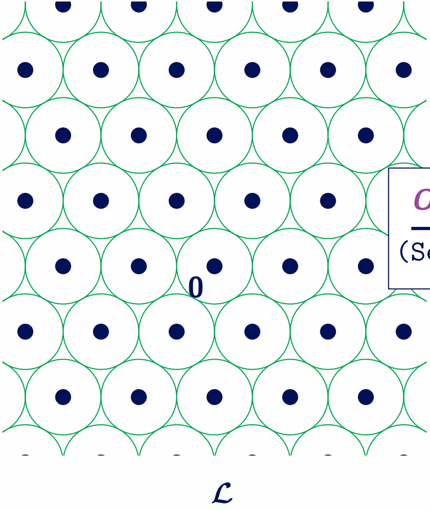



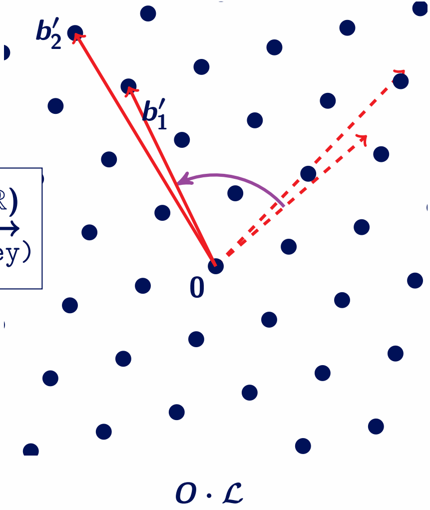

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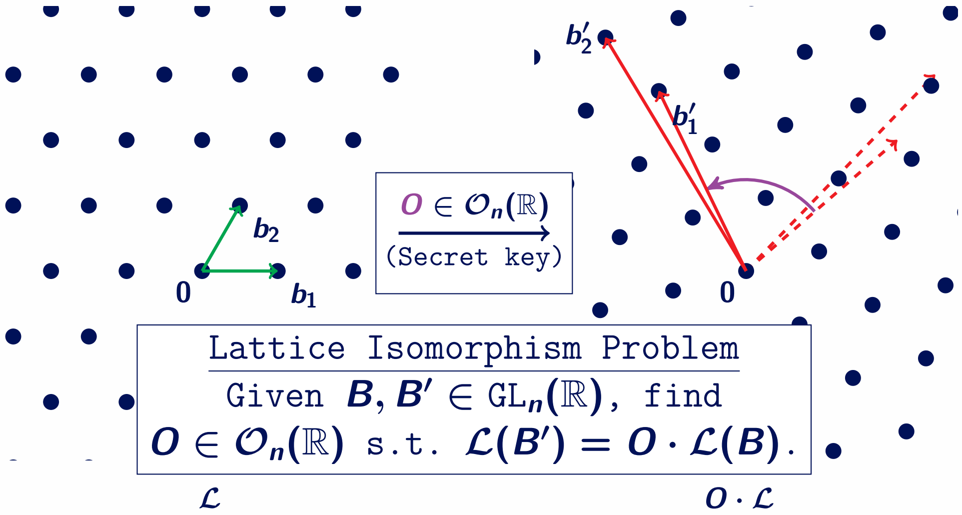


**Lattice Isomorphism Problem**

Given $B, B' \in \mathrm{GL}_n(\mathbb{R})$, find $O \in \mathcal{O}_n(\mathbb{R})$ s.t. $\mathcal{L}(B') = O \cdot \mathcal{L}(B)$.

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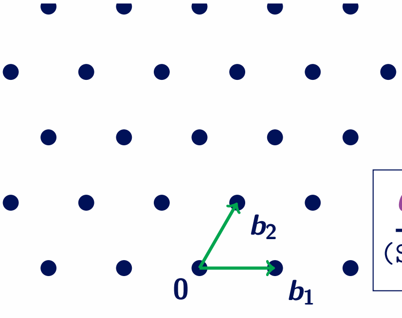
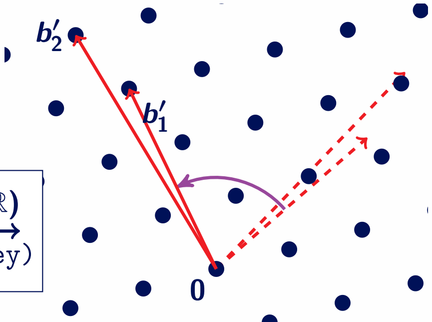



$O \in \mathcal{O}_n(\mathbb{R})$ ⟶ (Secret key)




**Lattice Isomorphism Problem**

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# Lattice Isomorphism Problem

**LIP**

Given $\boldsymbol{B}, \boldsymbol{B}' \in \mathrm{GL}_n(\mathbb{R})$ of isomorphic lattices, find $\boldsymbol{O} \in \mathcal{O}_n(\mathbb{R})$ and $\boldsymbol{U} \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $\boldsymbol{B}' = \boldsymbol{O} \cdot \boldsymbol{B} \cdot \boldsymbol{U}$.

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- The lattice analogue of 'vintage' McEliece $\boldsymbol{G}' = \boldsymbol{P} \cdot \boldsymbol{G} \cdot \boldsymbol{S}$,
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- The lattice analogue of 'vintage' McEliece $G' = P \cdot G \cdot S$,
- and Oil and Vinegar $\mathcal{P} = \mathcal{Q} \circ \mathcal{S}$.
- Best known attacks require to solve SVP.

$$B' = O \cdot B \cdot U \, .$$

$$\boldsymbol{B}' = \boldsymbol{O} \cdot \boldsymbol{B} \cdot \boldsymbol{U}.$$

Sidestep real values!

$\boldsymbol{O} \in \mathcal{O}_n(\mathbb{R})$

Sample $\boldsymbol{U} \in \mathrm{GL}_{\boldsymbol{n}}(\mathbb{Z})$ s.t.
$\boldsymbol{B}'$ is independent of $\boldsymbol{B}$.

$$\boldsymbol{B}' = \boldsymbol{O} \cdot \boldsymbol{B} \cdot \boldsymbol{U}.$$

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Lattices $\Longrightarrow$ Quadratic Forms

*Keep the geometry, forget the embedding.*

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LIP restated:

Find $U \in \mathrm{GL}_n(\mathbb{Z})$ s.t. $Q' = U^t Q U .$

# An average-case distribution

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$\Longrightarrow$ average-case LIP, ZKPoK and identification scheme.

$\Longrightarrow$ Worst-case to average-case reduction over $[Q]$.

Decodable lattice $\mathcal{L}$ $\Longrightarrow$ Encryption scheme.

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Decoding attack (BDD): $\mathbf{gap}(\mathcal{L}, \rho)$.

# Decodable Lattices

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|---|---|---|---|
| 'Random' Lattice | $\Theta(1)$ | $\Theta(1)$ | $2^{\Theta(n)}$ |
| Prime Lattice | $\Theta(\log n)$ | $\Omega(\sqrt{n})$ | $\Theta(\log n)$ [CR88, DP19] |
| Barnes-Sloane | $\Theta(\sqrt{\log n})$ | $\Omega(\sqrt{n})$ | $\Theta(\sqrt{\log n})$ [MP20] |
| Reed-Solomon | $\Theta(\sqrt{\log n})$ | $\Omega(\sqrt{n})$ | $\Theta(\sqrt{\log n})$ [BP22] |
| $\mathbb{Z}^n$ | $\Theta(\sqrt{n})$ | $\Theta(\sqrt{n})$ | $\Theta(\sqrt{n})$ |
| NTRU, LWE | $\Omega(1)\ldots O(\sqrt{n})$ | $\Omega(1)$ | $\Omega(\sqrt{n})$ |
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?: gaps $\leq$ poly-log$(n)$,

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$\mathbb{Z}^n$ seems enough to match security of LWE and NTRU.

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$\mathbb{Z}^n$ seems enough to match security of LWE and NTRU.

End goal: do even better.

# Conclusion

Any lattice $\implies$ Identification scheme.

Decodable lattice $\mathcal{L}$ $\implies$ Encryption scheme.

Gaussian sampleable lattice $\mathcal{L}$ $\implies$ Signature scheme.

$\mathbb{Z}^n$ seems enough to match
security of LWE and NTRU.

End goal: do even better.

Thanks! :)