A canonical form for positive definite matrices

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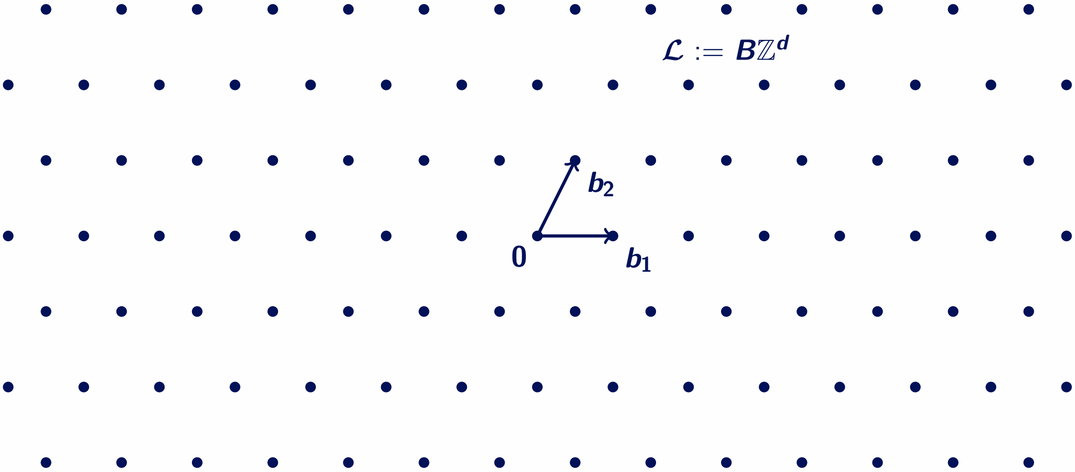
Lattice


$\mathcal{L} := \boldsymbol{B}\mathbb{Z}^{d}$
$\boldsymbol{b}_2$
$\boldsymbol{0}$
$\boldsymbol{b}_1$

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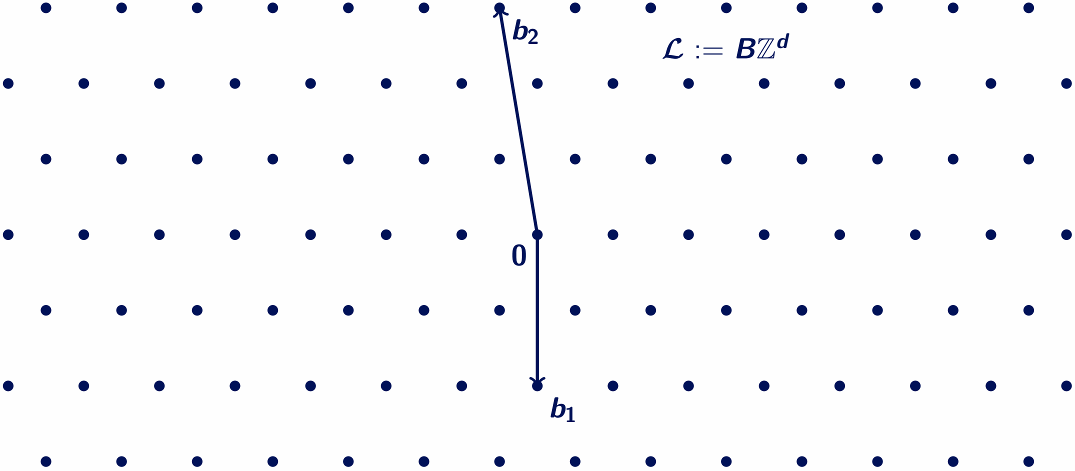

Lattice Equality

$$\mathcal{L}(\boldsymbol{B}_1) = \mathcal{L}(\boldsymbol{B}_2)$$

$$\iff$$

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- What if we have many bases $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_m$?
- $\boldsymbol{O}(\boldsymbol{m}^2)$ pairwise checks.

Hermite normal form

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- Polytime algorithm to compute HNF (using LLL to prevent coefficient blow-up).

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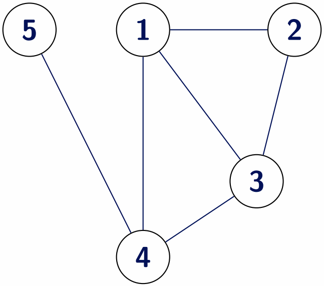
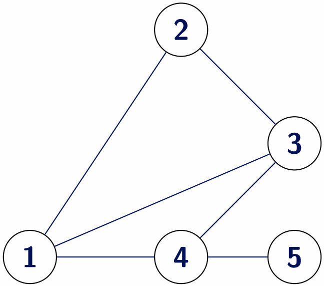
- Only $\boldsymbol{O}(\boldsymbol{m})$ queries/insertions in a hash table.
- Variant can be used for left action: $\mathrm{HNF}_{\boldsymbol{L}}(\boldsymbol{UB}^{\boldsymbol{t}}) = \mathrm{HNF}_{\boldsymbol{L}}(\boldsymbol{B}^{\boldsymbol{t}})$.

Graph Isomorphism

Graph $\boldsymbol{G} = (\boldsymbol{V} = [\boldsymbol{n}], \boldsymbol{E} \subset \boldsymbol{V} \times \boldsymbol{V})$



Graph $\boldsymbol{G'} = (\boldsymbol{V} = [\boldsymbol{n}], \boldsymbol{E'})$



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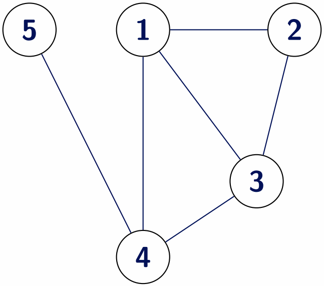

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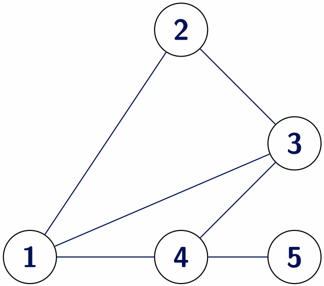

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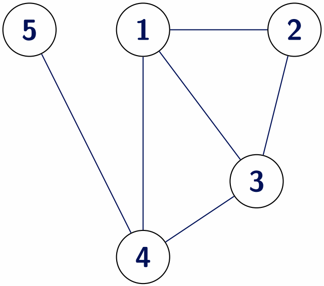

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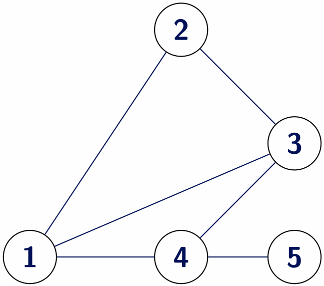

- Graph equality: $E = E'$.
- Graph Automorphisms: $\text{Stab}(G) = \{\sigma \in \text{Sym}_{|V|} : \sigma(E) = E\}$.

$$\sigma(E) := \{(\sigma(i), \sigma(j)) : (i, j) \in E\}$$

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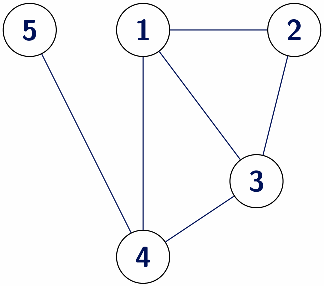

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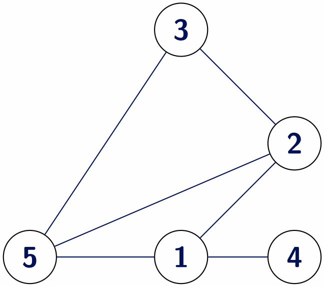
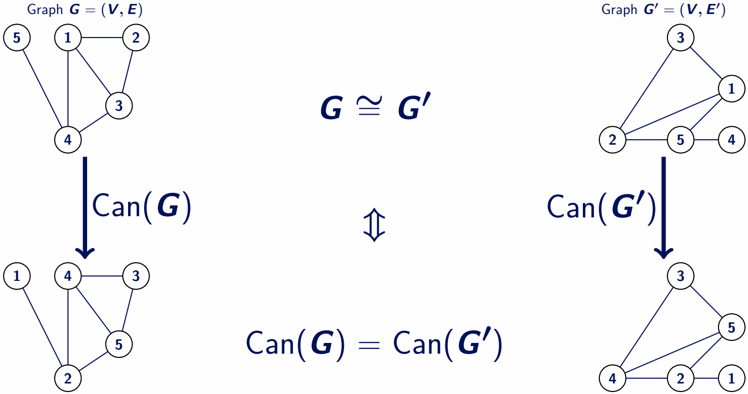


- Graph Isomorphism: $G \cong G' \Leftrightarrow \sigma(E) = E'$ for some $\sigma \in \mathrm{Sym}_n$.

Canonical Graph Ordering

- Give a vertex ordering of the graph purely based on the graph structure.



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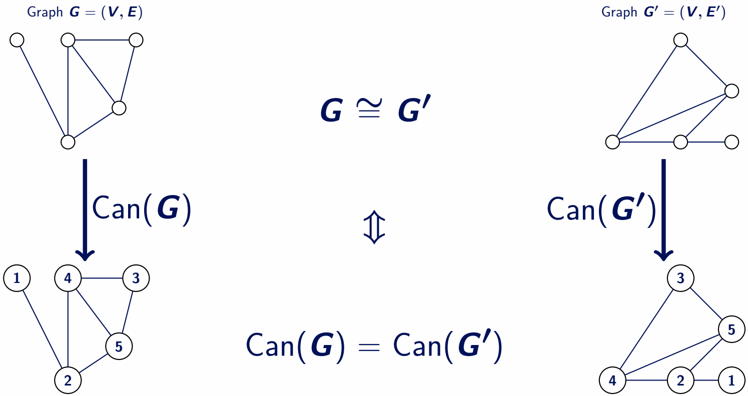

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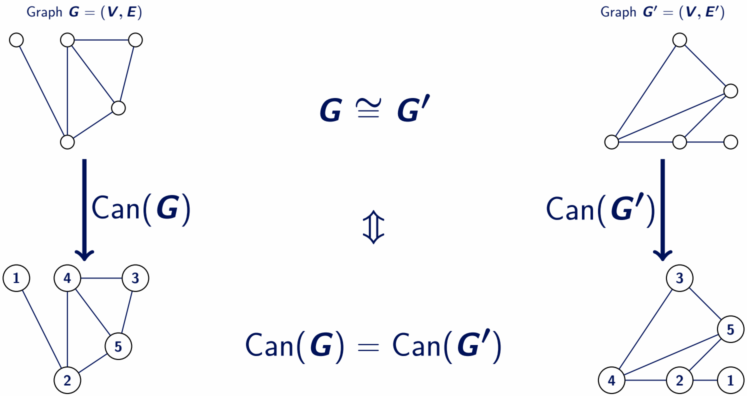

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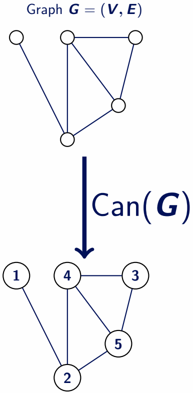
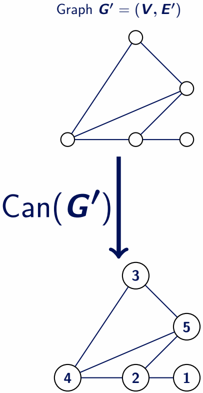
- Permutation (relative to input) is unique up to $\mathrm{Stab}(\boldsymbol{G})$.



$$G \cong G'$$

$$\Updownarrow$$

$$\mathrm{Can}(G) = \mathrm{Can}(G')$$



Complexity

- More generally for weighted complete graphs $\boldsymbol{G}$ with weights $\boldsymbol{W} = (\boldsymbol{w_{ij}})_{\boldsymbol{ij}}$:

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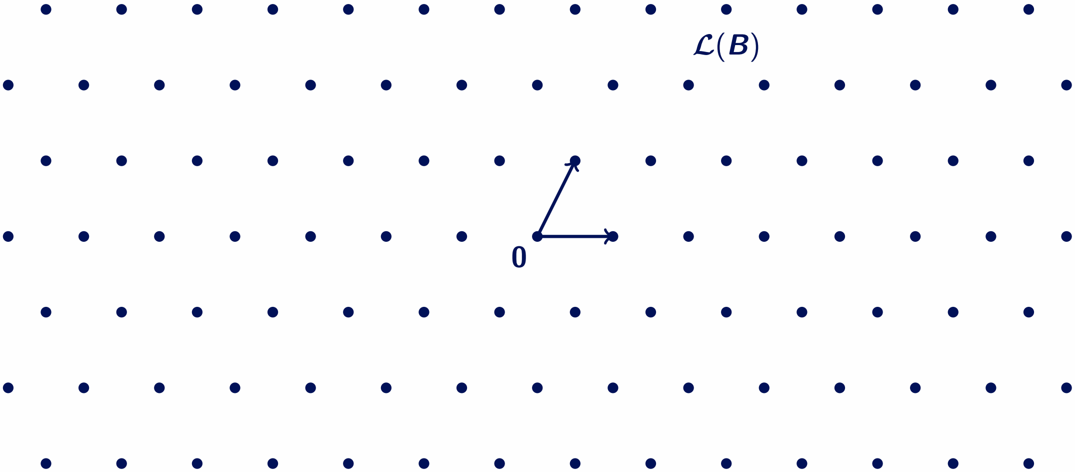

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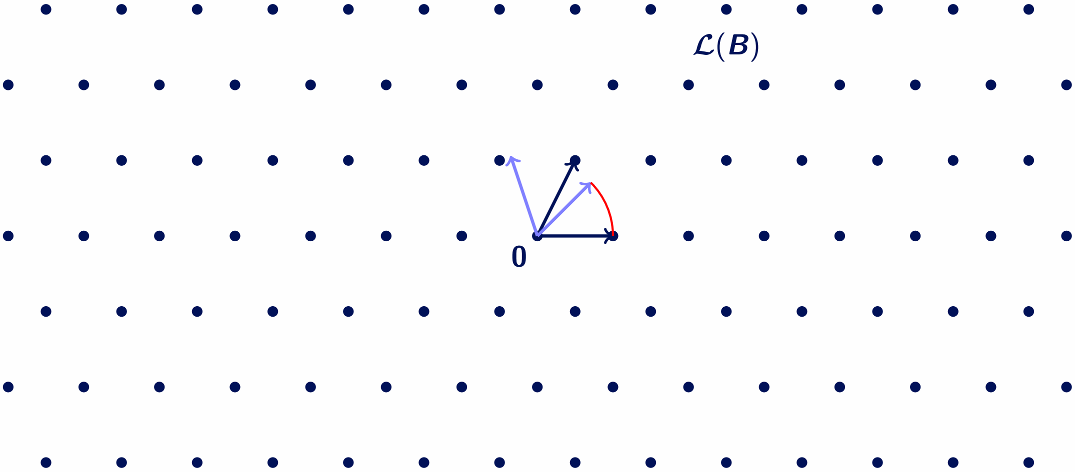

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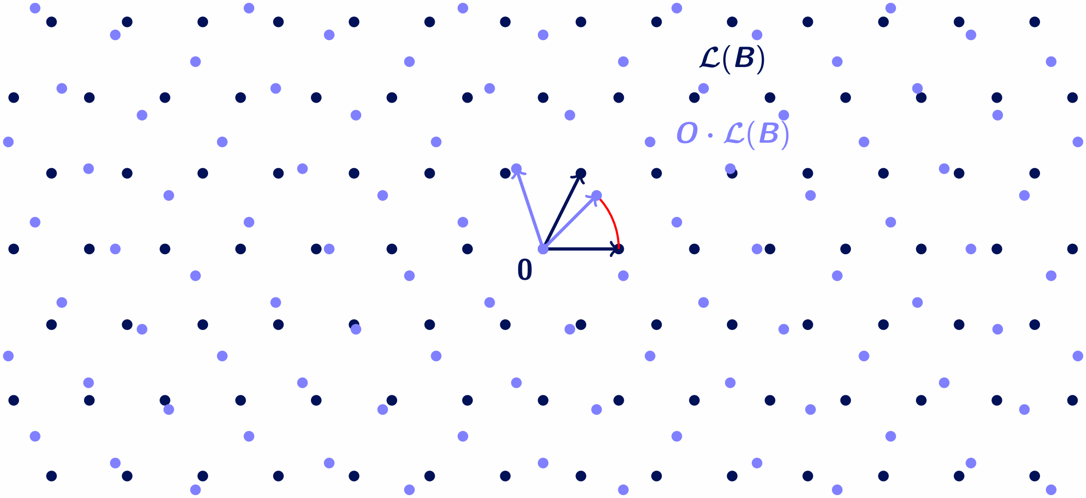

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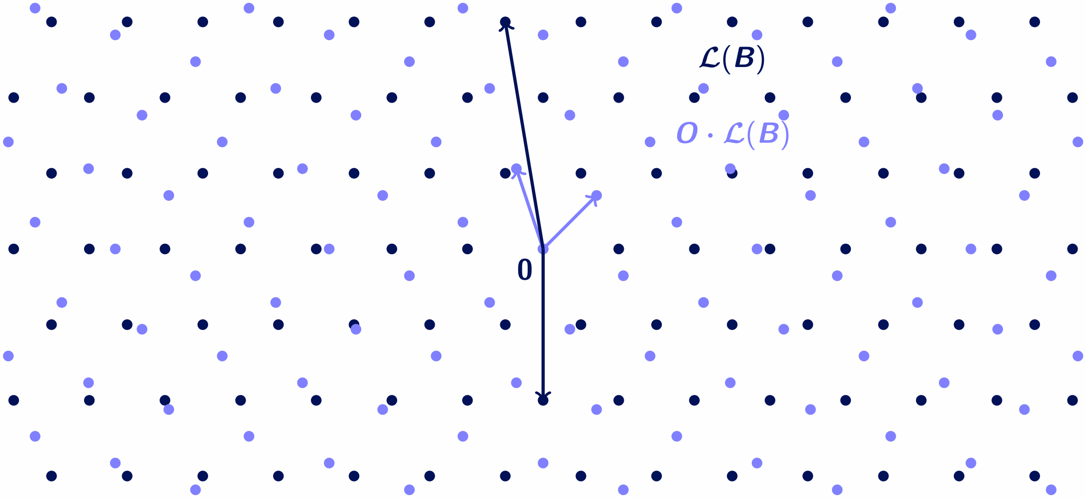

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- If either $\boldsymbol{O}$ or $\boldsymbol{U}$ is trivial: linear algebra.
- Use $\boldsymbol{O}^t \boldsymbol{O} = \boldsymbol{I}$ to remove the orthonormal transformation.

Quadratic Forms

- The gram matrix $\boldsymbol{A} = \boldsymbol{B}^t \boldsymbol{B} \in \mathcal{S}^d_{>0}$ induces a quadratic form:

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  (1) $\mathrm{Can}(\boldsymbol{A}) \sim \boldsymbol{A}$.
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- Can we construct a canonical function $\mathrm{Can} : \mathcal{S}^d_{>0} \to \mathcal{S}^d_{>0}$ such that
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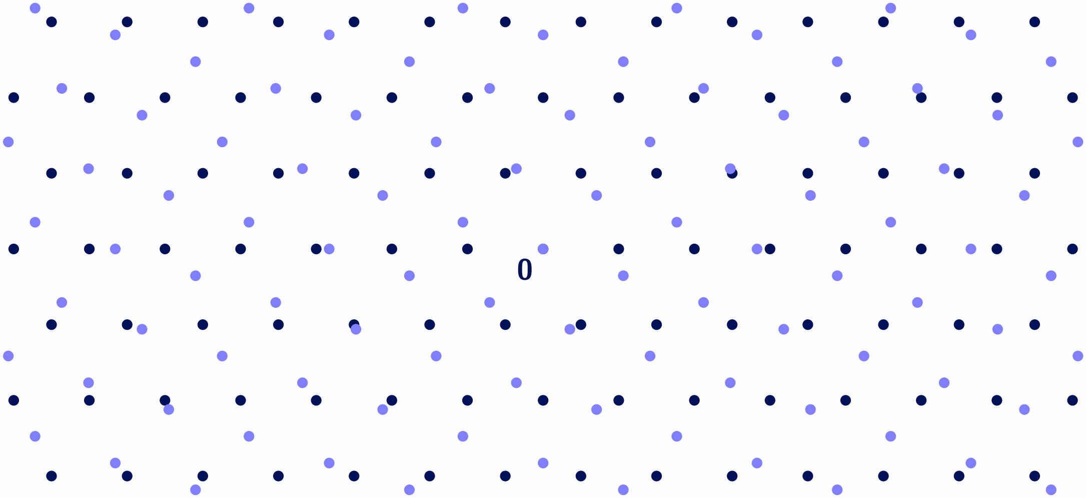
Characteristic Vector Set



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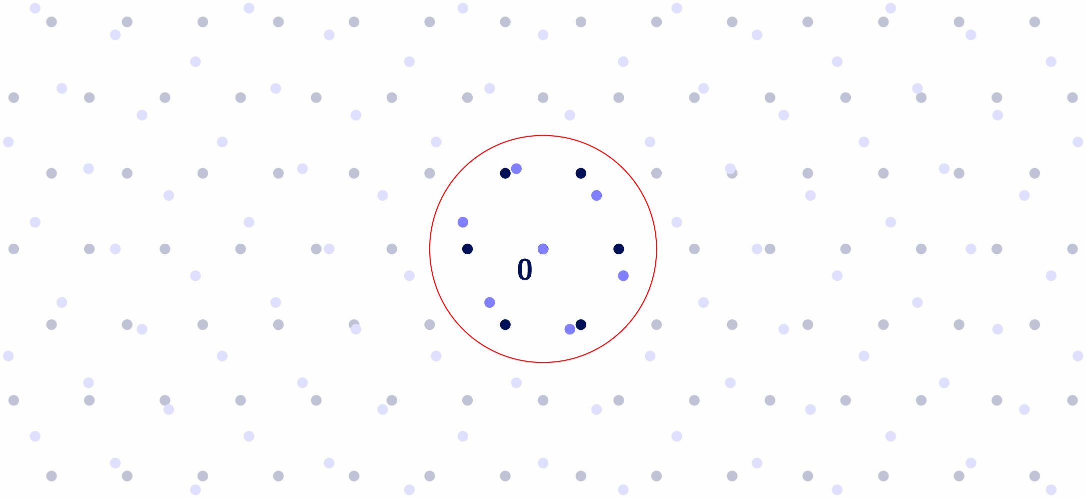

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- Used by W. Plesken and B. Souvignier (1997) to compute lattice automorphisms and isomorphisms.

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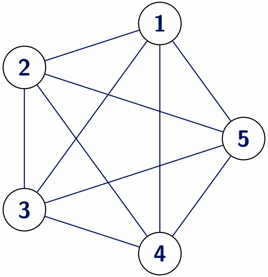
- $v_i^t A_1 v_j = w_i^t A_2 w_j$ for all i, j is also *sufficient* for such a U to exist.
- $\mathcal{V}(A_2) = \{w_1, \ldots, w_n\}$.
- We want to find a permutation σ such that $v_i A_1 v_j = w_{\sigma(i)} A_2 w_{\sigma(j)}$ for all i, j.

Back to Graph Isomorphism

$G(\mathcal{V}(A_1))$



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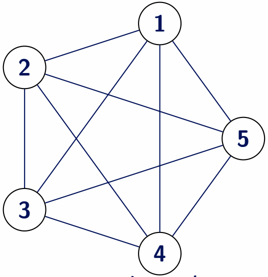

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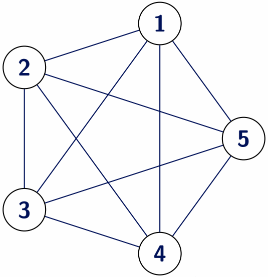

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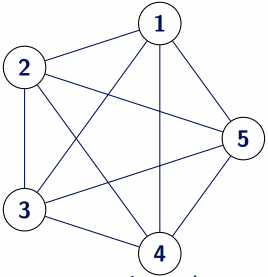

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- $\mathrm{Stab}(A_i) \cong \mathrm{Stab}(G(\mathcal{V}(A_i)))$.

Canonical Matrix

- From the graph we obtain some canonical ordering of $\mathcal{V}(\boldsymbol{A}) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, say

$$\begin{bmatrix} \vdots & \vdots & & \vdots & \vdots \\ \boldsymbol{v}_{23} & \boldsymbol{v}_{16} & \cdots & \boldsymbol{v}_{3} & \boldsymbol{v}_{7} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix} \in \mathbb{Z}^{d \times n}$$

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- Now we can apply HNF: $A_1 \sim A_2 \iff \mathrm{HNF}_L(M(A_1)) = \mathrm{HNF}_L(M(A_2))$

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- Then we have:

$$\begin{aligned} \mathrm{Can}(\boldsymbol{U^t AU}) &= \boldsymbol{T^t_{U^t AU}}(\boldsymbol{U^t AU})\boldsymbol{T_{U^t AU}} \\ &= \boldsymbol{T_A U^{-t} U^t A U U^{-1} T_A} = \boldsymbol{T_A^t A T_A} = \mathrm{Can}(\boldsymbol{A}) \end{aligned}$$

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- For $\boldsymbol{F} = \mathbb{Q}$ these operations are polynomially bounded in the input size of $\boldsymbol{A}$.

Practical Complexity

- Efficient in practice.

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Type	Samples	n	min	avg	max	min	avg	max
Perfect	10 963	2–8	0.00041	0.0032	0.086	6	73.74	240
	524 288	9	0.0039	0.00594	0.11	90	94.04	272
Random	100	10	0.0015	0.08	2.03	20	100.36	988
	100	20	0.016	0.17	4.18	40	114.34	812
	100	30	2.43	23.41	511.42	60	93.46	310
	100	40	5.18	24.91	251.51	82	107.7	240
Catalogue	107	2-16	0.00018	2.12	36.71	4	630.47	4320

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- Often a critical bottleneck in enumeration algorithms (Kneser's method, Voronoi's algorithm).
- Enumeration of forms of bounded discriminant, or small (spinor) class number.
  - Brandt–Intrau tables of reduced ternary forms with discriminant $\leq \mathbf{1000}$
  - Nipp's tables of positive definite primitive quaternary forms with discriminant $\leq \mathbf{1732}$.
  - Complete table of lattices with class number $\mathbf{1}$ due to Kirschmer–Lorch
- Perfect form enumeration to solve the lattice packing theorem.
  - Estimates are in the order of a **billion** perfect forms in dimension $\mathbf{9}$.
  - Canonical form computed in an average of $\mathbf{6}$ milliseconds.
- Algebraic Modular Forms related to Kneser's method.

Conclusions

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- It is efficient in practice and has many applications.

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