The lattice packing problem in dimension 9 by Voronoi's algorithm

Mathieu Dutour Sikirić & **Wessel van Woerden** (PQShield).

université de BORDEAUX

PQ SHIELD

Sphere Packing Problem

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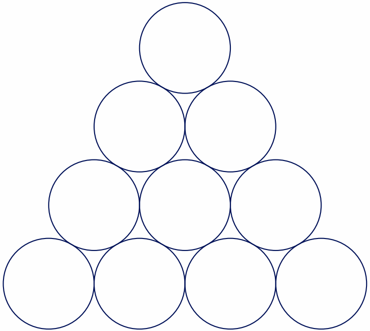

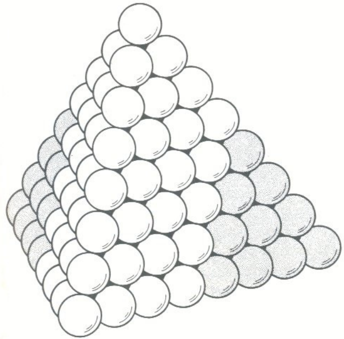

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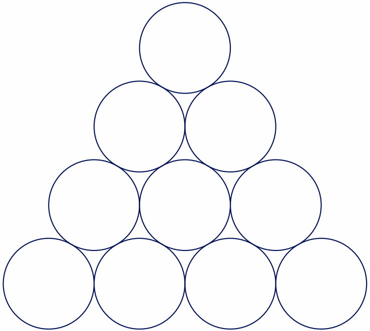

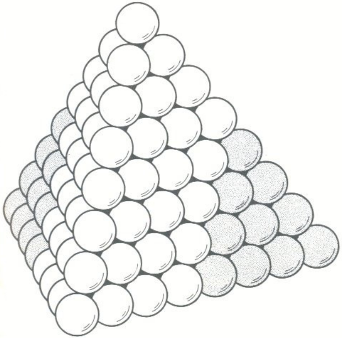

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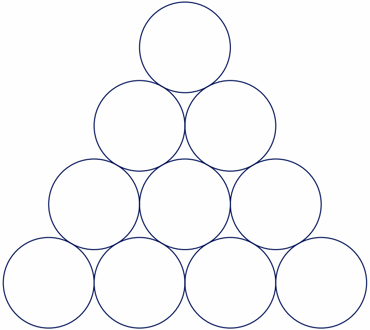

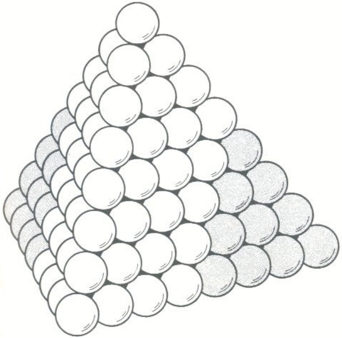

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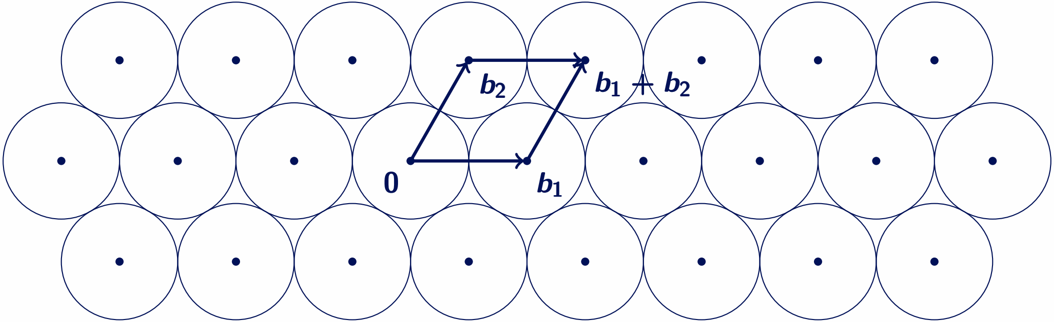
- Dimension **3** only in **1998** by a computational proof (Thomas Hales)

Lattice Packing Problem



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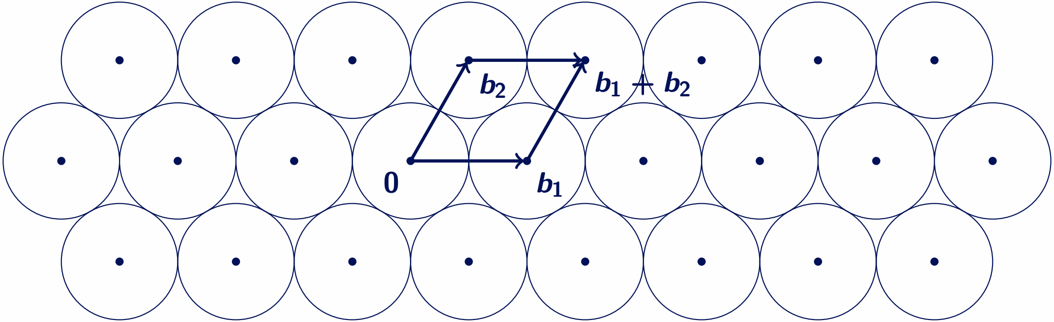

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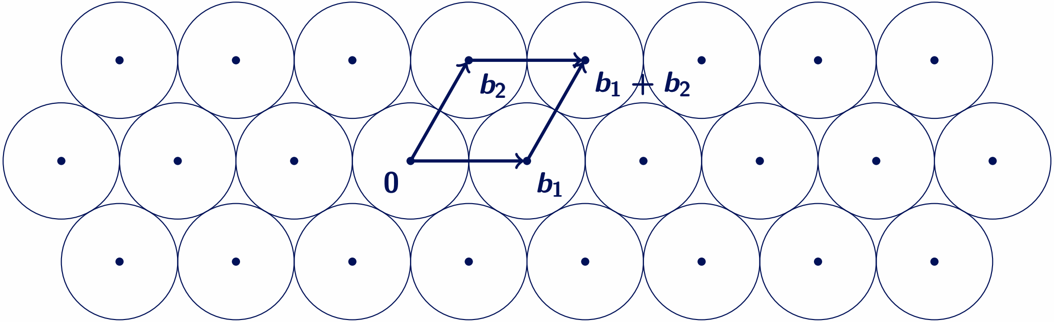

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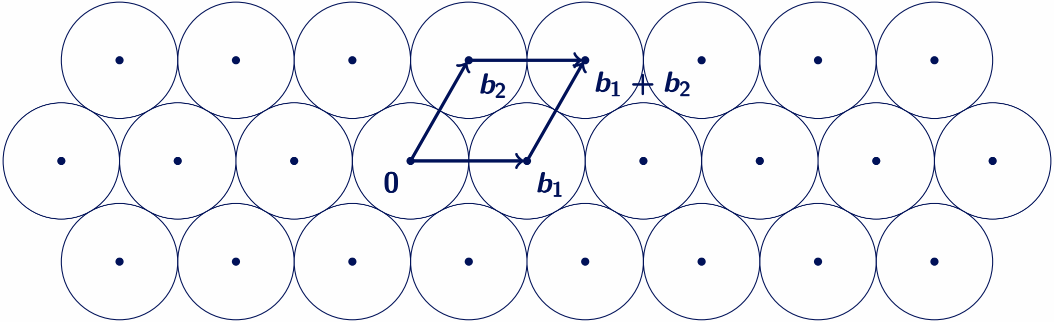

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- What about dimension $\mathbf{9}$?

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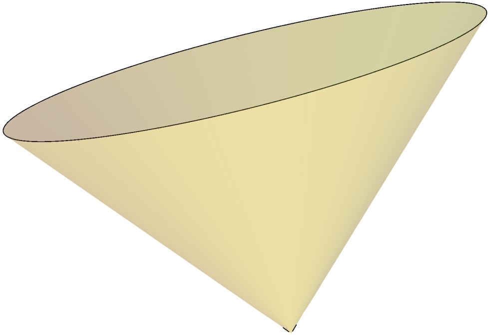
- **Corollary:** the laminated lattice $\boldsymbol{\Lambda_9}$ is the unique densest lattice packing.

Solution space



- Represent $\boldsymbol{L} = \boldsymbol{B} \cdot \mathbb{Z}^{\boldsymbol{d}}$ by its **positive definite gram matrix** $\boldsymbol{Q} := \boldsymbol{B}^t \boldsymbol{B}$.
- Cone of positive definite matrices

$$\mathcal{S}^{d}_{>0} \subset \mathcal{S}^{d} \subset \mathbb{R}^{d \times d}.$$

$$\dim(\mathcal{S}^d) = \tfrac{1}{2} d(d+1) =: n$$

- inner product: (to show these pictures)

$$\langle A, B \rangle := \mathrm{Tr}(A^t B) = \sum_{i,j} A_{ij} B_{ij}$$

- $Q \in \mathcal{S}^d$ defines a quadratic form by

$$Q[x] := x^t Q x = \langle Q, x x^t \rangle \ \ \forall x \in \mathbb{R}^d$$

Hermite Constant

- Lattice $\boldsymbol{L} = \boldsymbol{B} \cdot \mathbb{Z}^{\boldsymbol{d}} \implies$ PQF $\boldsymbol{Q} = \boldsymbol{B}^{\boldsymbol{t}} \boldsymbol{B} \in \boldsymbol{S}^{\boldsymbol{d}}_{>\mathbf{0}}$.

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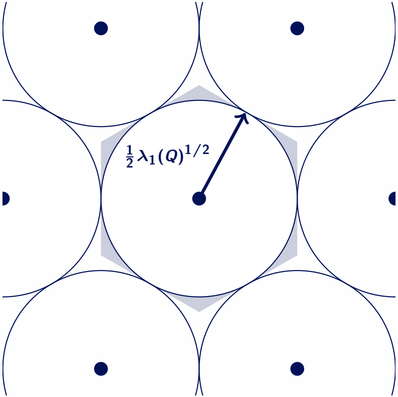
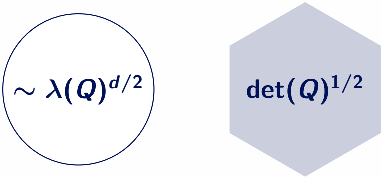


$$\lambda(Q) := \min_{x \in \mathbb{Z}^d \setminus \{0\}} Q[x] = \min_{y \in L \setminus \{0\}} \|y\|^2$$

$$\operatorname{Min} Q := \{x \in \mathbb{Z}^d : Q[x] = \lambda(Q)\}$$



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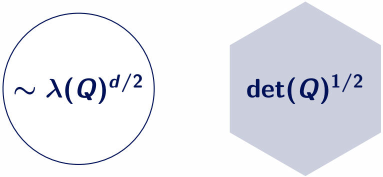

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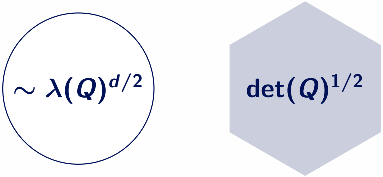

- Hermite invariant:

$$\gamma(Q) = \frac{\lambda(Q)}{(\det Q)^{1/d}} \sim \text{density}(L)^{2/d}$$

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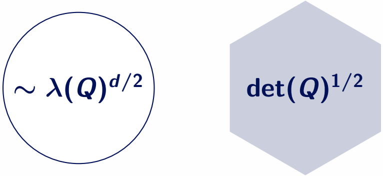

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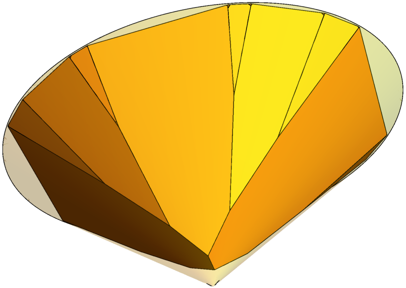
- Lattice packing problem $\Leftrightarrow$ determine Hermite's constant:

$$\gamma_d := \sup_{Q \in \mathcal{S}^d_{>0}} \gamma(Q)$$

Ryshkov Polyhedra

- For $\lambda > 0$ we define the Ryshkov Polyhedra

$$\mathcal{P}_\lambda = \{Q \in \mathcal{S}^d_{>0} : \lambda(Q) \geq \lambda\}$$



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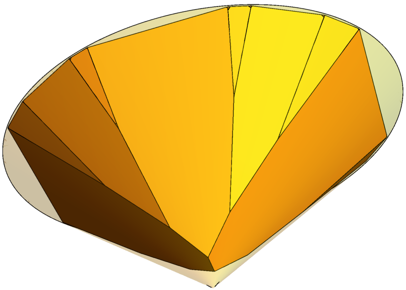

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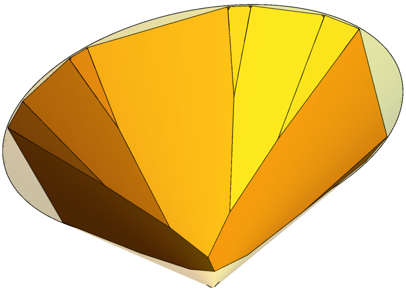

- Each facet corresponds to some primitive $\pm x \in \mathbb{Z}^d$.
- Locally finite

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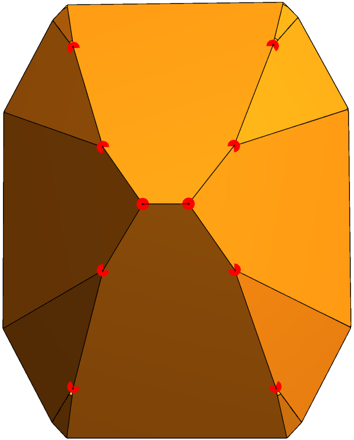
- Minkowski: $\det(Q)^{1/d}$ is (strictly) concave on $\mathcal{S}^d_{>0}$

  $\Longrightarrow$ Local optima at vertices of $\mathcal{P}_\lambda$. (uses that $\mathcal{P}_\lambda$ is locally finite)



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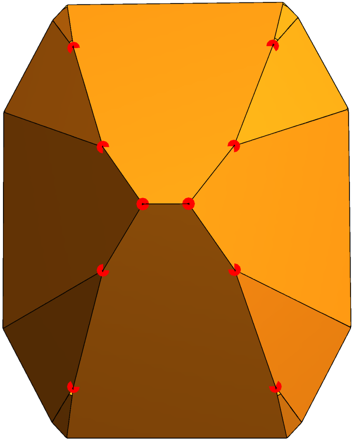
  $\Leftrightarrow$ Q is fully determined by Min Q and $\lambda_1(Q)$.

Perfect forms



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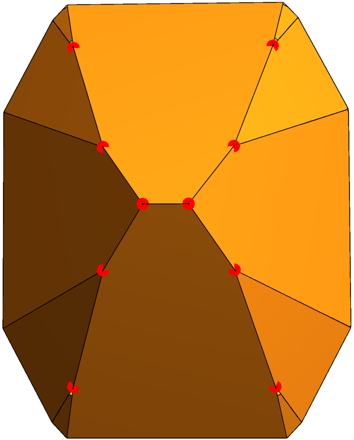

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- Voronoi's algorithm: enumerate all perfect forms (up to equivalence/similarity)

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- We have $\mathrm{Min}\, U^t Q U = U^{-1} \cdot \mathrm{Min}\, Q$. ($\mathrm{GL}_d(\mathbb{Z})$ acts on $\mathcal{P}_\lambda$)

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In practice..

d	# p_d
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3	1 (Gauss, 1840)
4	2 (Korkine & Zolotarev, 1877)
5	3 (Korkine & Zolotarev, 1877)
6	7 (Barnes, 1957)
7	33 (Jaquet, 1993)
8	10916 (DSV, 2005)
9	$\geq$ 500.000 (DSV, 2005)
	$\geq$ 23.000.000 (**vW**, 2018)

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	Many more, to be continued...

Voronoi's Algorithm
Challenges & Solutions

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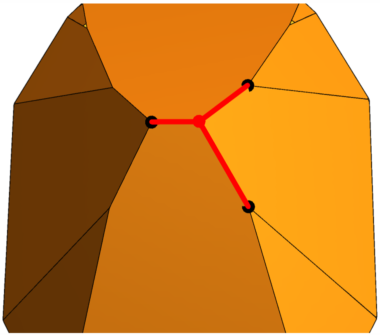

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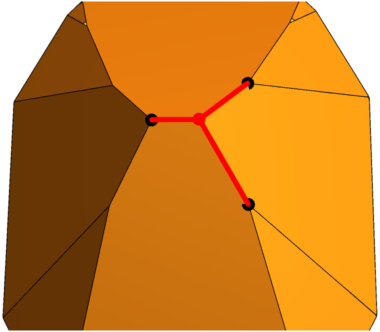

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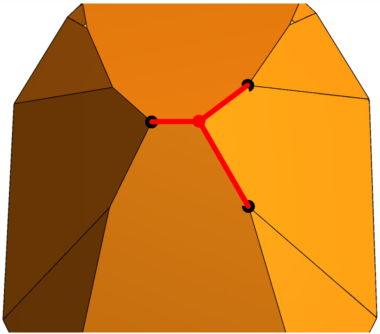

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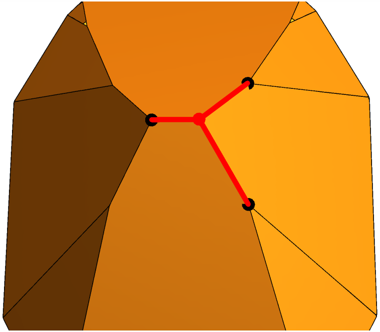

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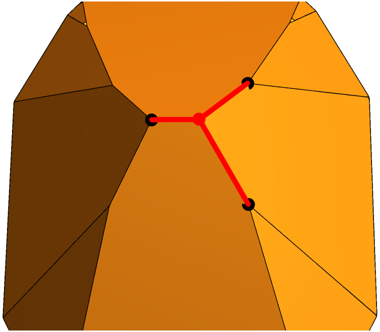

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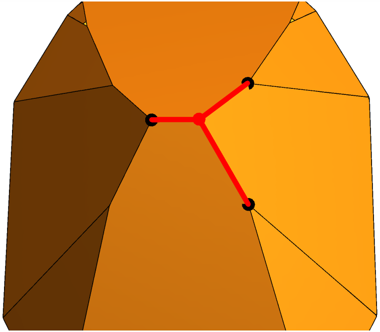

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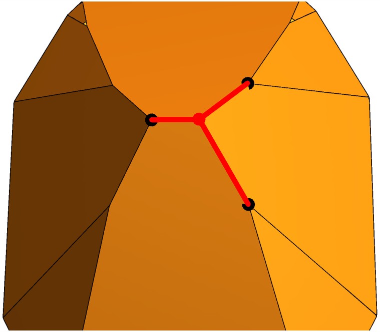

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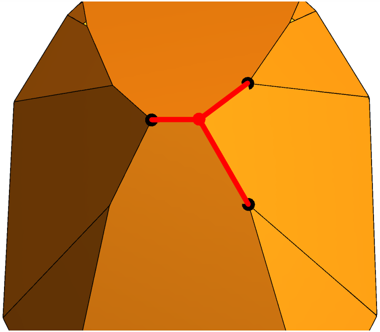

Canonical functions

- Group $\boldsymbol{G}$ acting on a set $\boldsymbol{X}$, orbit equality determines an equivalence relation on $\boldsymbol{X}$.
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- Used for: **PQF**, **face** and **polyhedral** equivalence.

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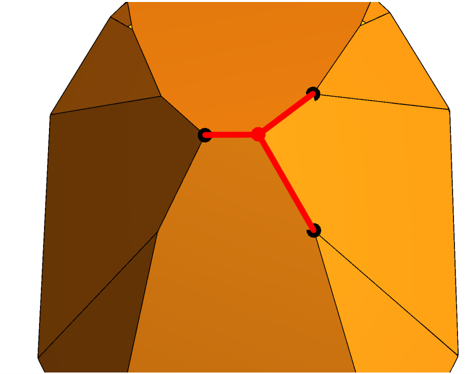
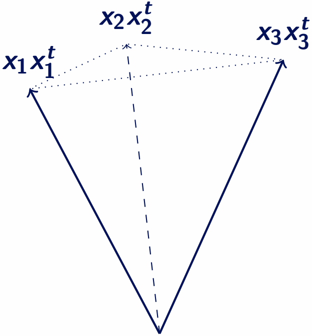
- **Details:** A canonical form for positive definite matrices. [ANTS 2020, **DS**HV**v**W]

Dual Description Problem

- A (pointed) polyhedral cone $\mathcal{C} \subset \mathbb{R}^n$ can either be given by **facet inequalities** or by **extreme rays**.





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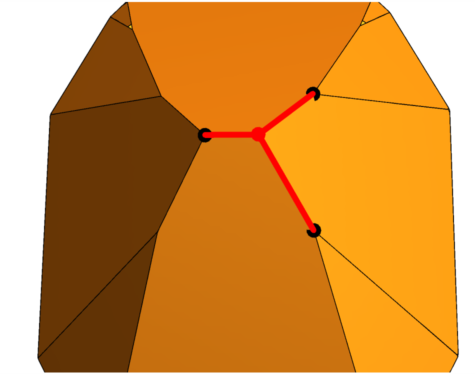

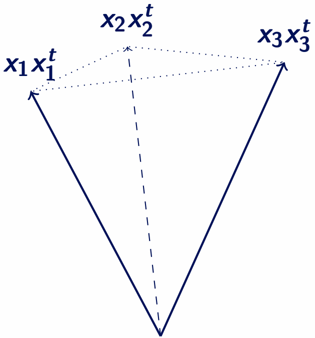

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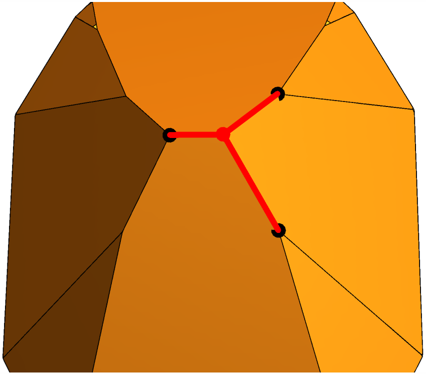
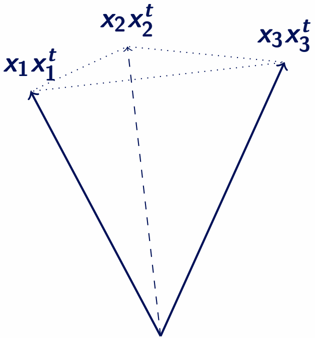
- The two directions are equivalent by duality.





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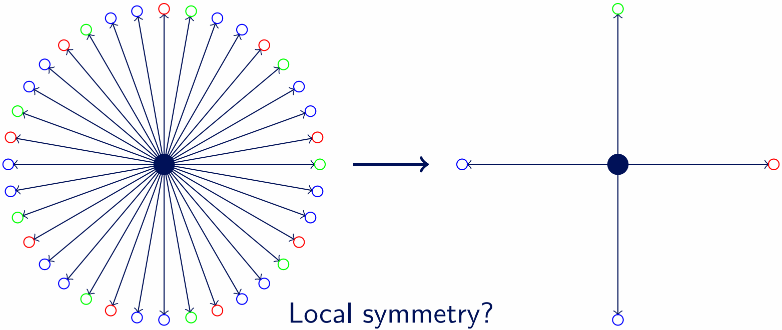
- $\mathcal{P}(\boldsymbol{Q}_{E_8})$: **120** facets in **36** dimensional space: **25.075.566.937.584** extreme rays...
- Many rays point to equivalent forms: $\boldsymbol{Q} + \alpha_1 \boldsymbol{R}_1 \sim \boldsymbol{Q} + \alpha_2 \boldsymbol{R}_2$



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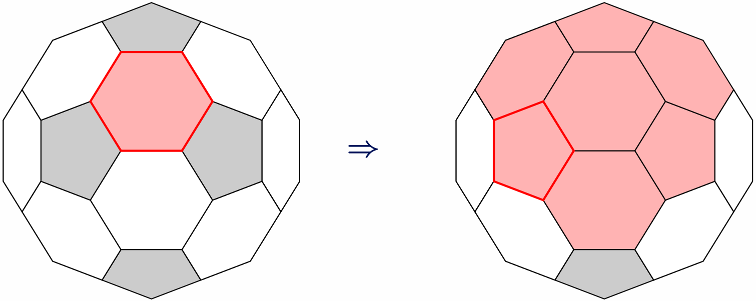
- **Even harder:** $\mathcal{P}(\boldsymbol{Q}_{\boldsymbol{\Lambda}_9})$ has **136** facets in a **45**-dimensional space.

Adjacency Decomposition Method



- Two $\boldsymbol{k}$-dimensional faces $\boldsymbol{F_1}, \boldsymbol{F_2}$ are **adjacent** if $\mathbf{dim}(\boldsymbol{F_1} \cap \boldsymbol{F_2}) = \boldsymbol{k} - \mathbf{1}$.

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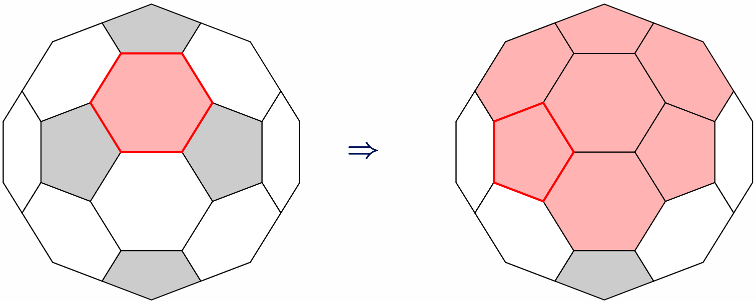

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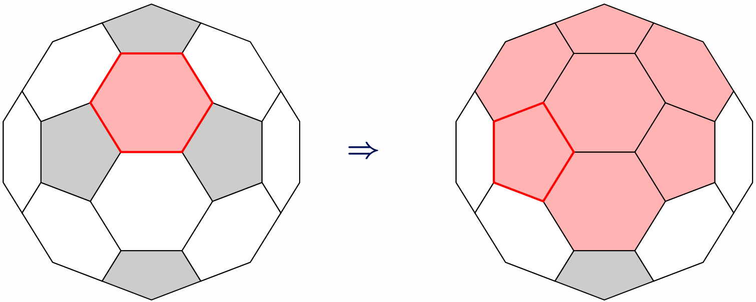

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- $\{\boldsymbol{F_2}$: adjacent to $\boldsymbol{F_1}\} \leftrightarrow \{$facets $\boldsymbol{H}$ of $\boldsymbol{F_1}\}$ $\qquad (\boldsymbol{H = F_1 \cap F_2})$.

Adjacency Decomposition Method

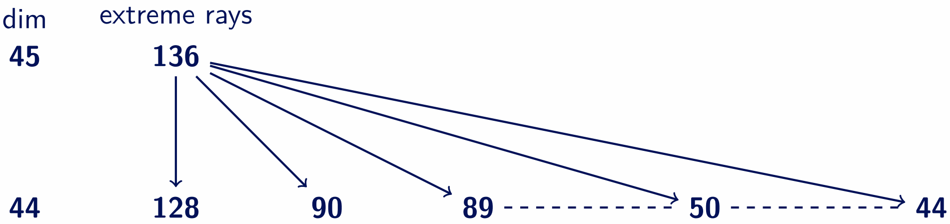
- Best explained in dual setting: $\mathcal{C} = \text{cone}([y_1, \ldots, y_m\} \subset \mathbb{R}^n$ with $G \subset \text{Aut}(\mathcal{C})$.

Algorithm: Adjacency Decomposition Method

1. Find at least one facet F.
2. Determine facets $H_1, \ldots, H_k$ of F, .i.e. ridges of $\mathcal{C}$ contained in F.
3. For all i
   - compute facet F_i of $\mathcal{C}$ such that $H_i = F \cap F_i$.
   - Keep F_i if G-inequivalent to all found facets.
4. Repeat **(2)** and **(3)** for each new facet.

- Step **(2)** is again Dual Description problem but dimension $n - 1$ and only with **extreme rays contained in F**.
- If still difficult, **recurse**: $G' = \text{Stab}(G, F)$.

Recursive Adjacency Decomposition Method



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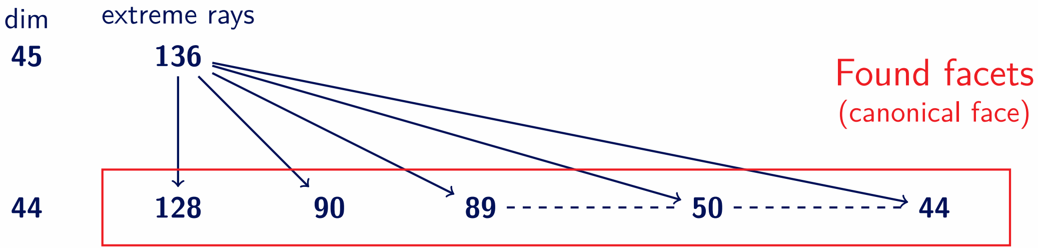

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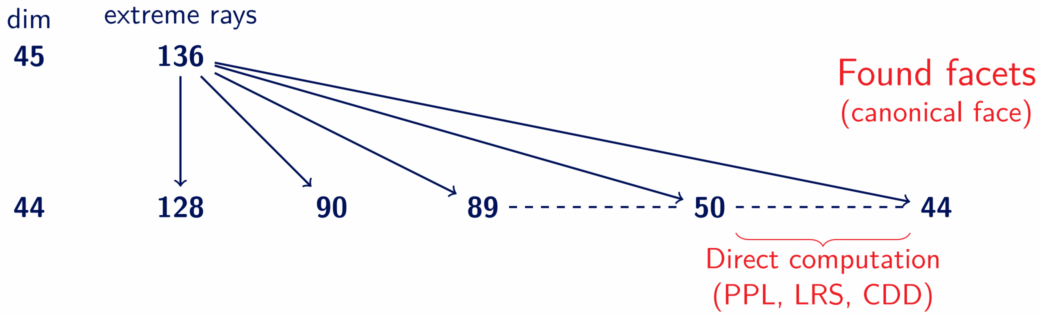

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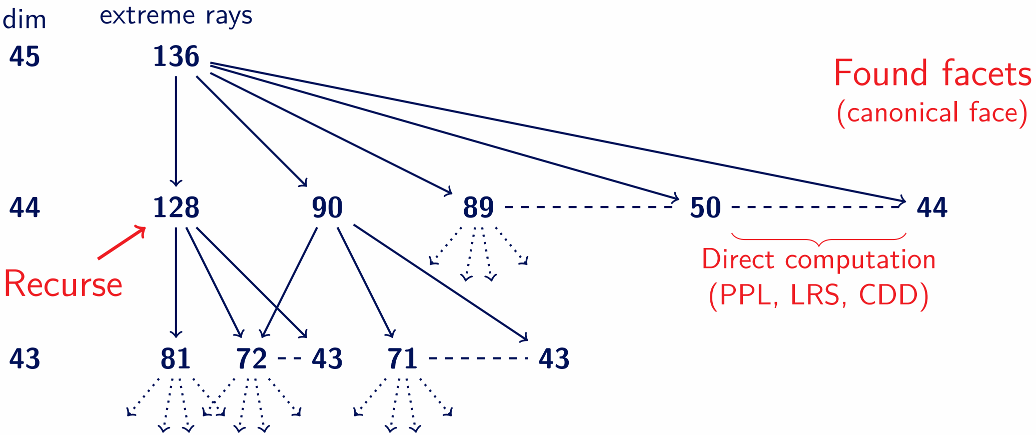

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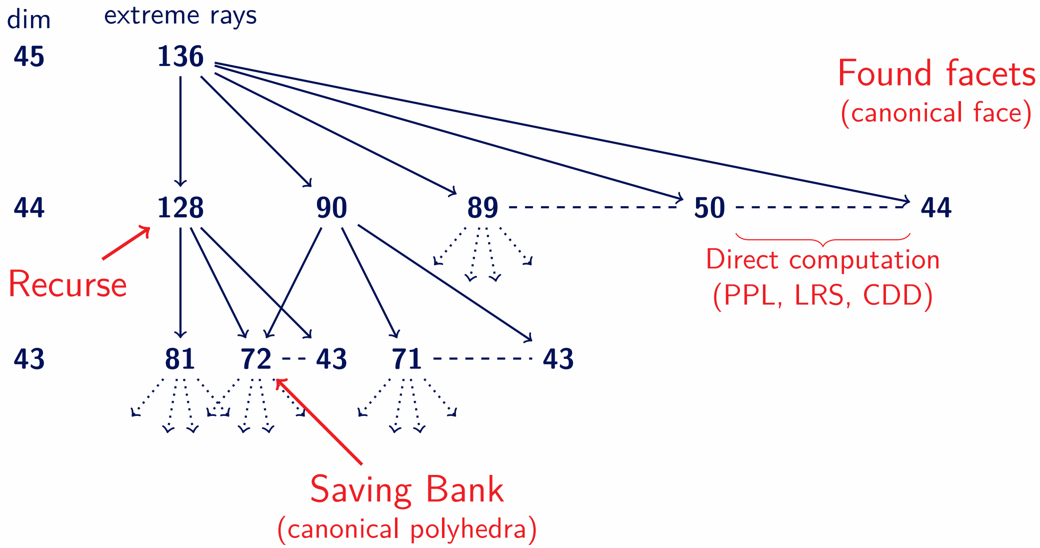

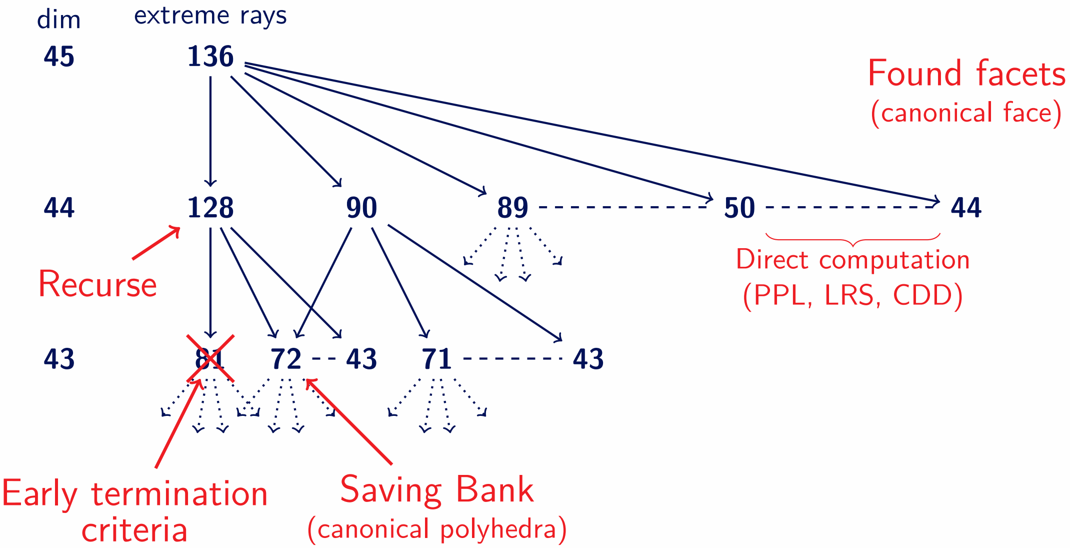
dim
extreme rays
45
136
Found facets
(canonical face)
44
128
90
89
50
44
Direct computation
(PPL, LRS, CDD)
Recurse
43
81
72
43
71
43
Early termination
criteria
Saving Bank
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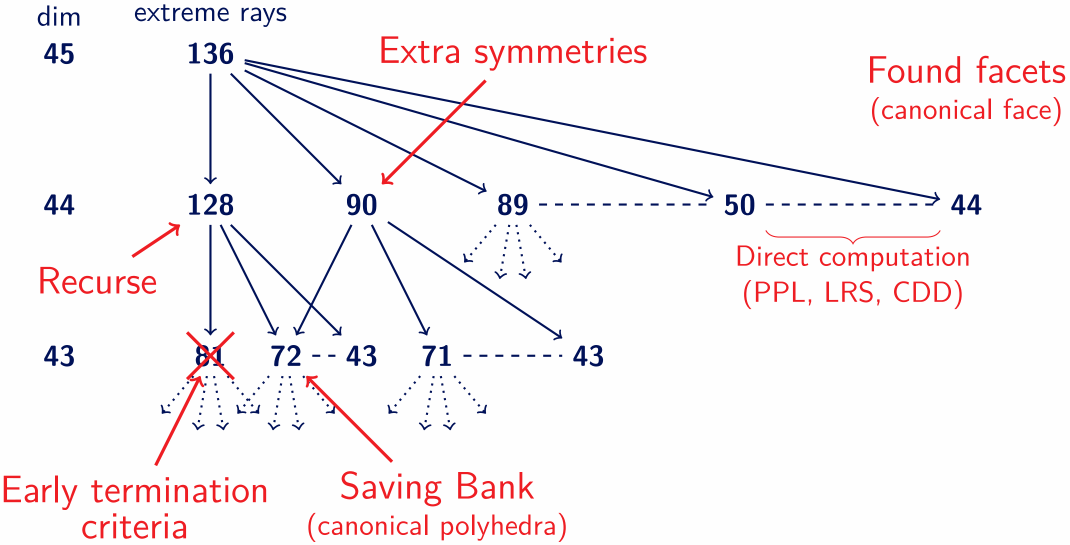

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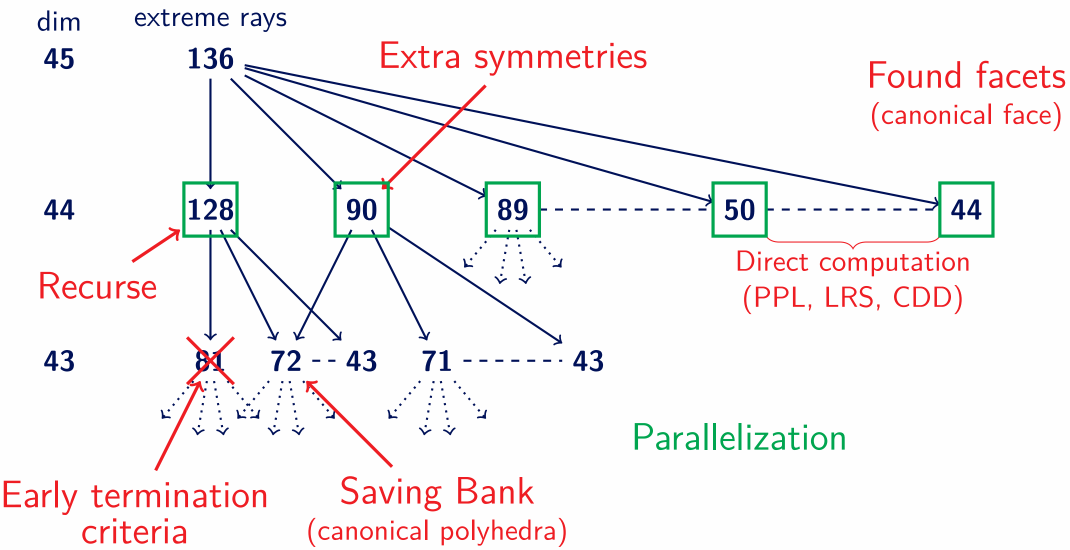

Results

Lattice packing problem in dimension 9

8 years and **$\pm$ 3 000 000 core hours later...**

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Theorem: Kissing numbers

The set of possible kissing numbers $|\mathrm{Min}\,(L)|$, for a lattice $L \subset \mathbb{R}^9$ of dimension **9**, is $2 \cdot \{1, \ldots, 91, 99, 120, \ldots, 129, 136\}$.

All perfect forms by their kissing number

$\lvert\min(Q)\rvert/2$	#	$\lvert\min(Q)\rvert/2$	#	$\lvert\min(Q)\rvert/2$	#
45	1 353 947 672	**61**	2 244	**77**	1
46	471 756 975	**62**	1 713	**78**	1
47	267 588 732	**63**	641	**79**	2
48	84 473 357	**64**	634	**80**	12
49	37 278 163	**65**	236	**81**	3
50	13 324 560	**66**	203	**82**	4
51	5 299 974	**67**	172	**84**	2
52	2 009 292	**68**	74	**85**	2
53	903 943	**69**	44	**88**	1
54	366 796	**70**	42	**90**	2
55	155 182	**71**	26	**91**	1
56	78 919	**72**	21	**99**	1
57	31 113	**73**	7	**129**	1
58	17 207	**74**	3	**136**	1
59	8 231	**75**	4		
60	4 820	**76**	6		

All perfect forms by their kissing number

99.9991% of all forms

$<$ **5%** of runtime.

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Table: Cost of dual description cases with more than **50**k core hours. These cases account for **1.5** million of the total amount of **2** million core hours spent on dual description instances.

$\|\min(Q)\|/2$	Core hours	$\|\text{linaut}(P)\|$	rays (orbits)	$\|\text{aut}(Q)\|$	neighbours (orbits)
136	59 277	660 602 880	64 001 686	10 321 920	1 038 153 863
84	75 467	12 288	171 496 157	384	1 514 557 045
99	84 197	589 824	137 739 671	18 432	1 842 205 495
90	85 349	73 728	185 824 962	2 304	2 058 568 310
74	95 784	128	333 146 387	16	1 257 559 244
80	97 118	7 680	108 828 919	480	764 775 430
81	181 570	1 296	254 734 260	2 592	254 734 260
80	219 437	128	772 745 513	256	772 745 513
82	245 030	432	680 747 757	864	680 747 757
76	355 554	24	1 549 616 491	48	1 549 616 491

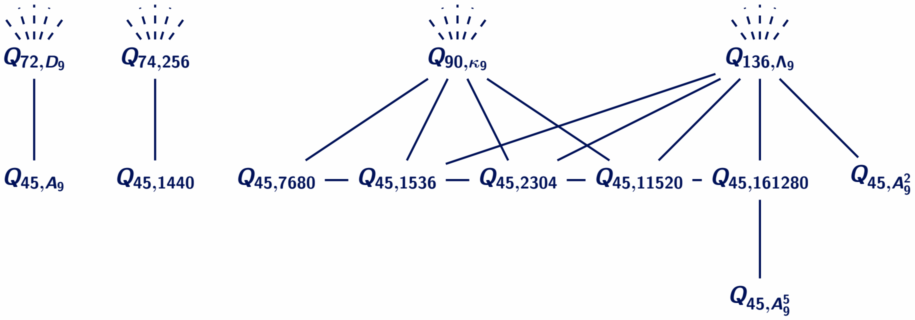


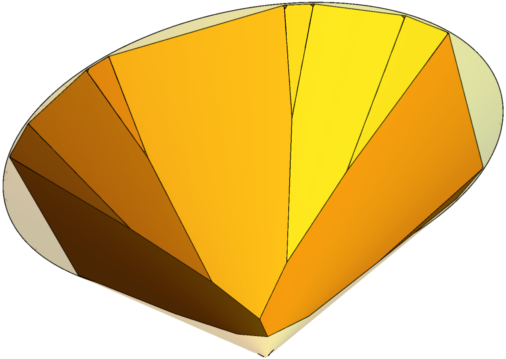
Figure: Part of Voronoi graph showing all perfect forms that are only connected via high-incidence perfect forms.

- All other forms are connected via forms with $|\mathrm{Min}\, Q| \leq 2 \cdot 58$.

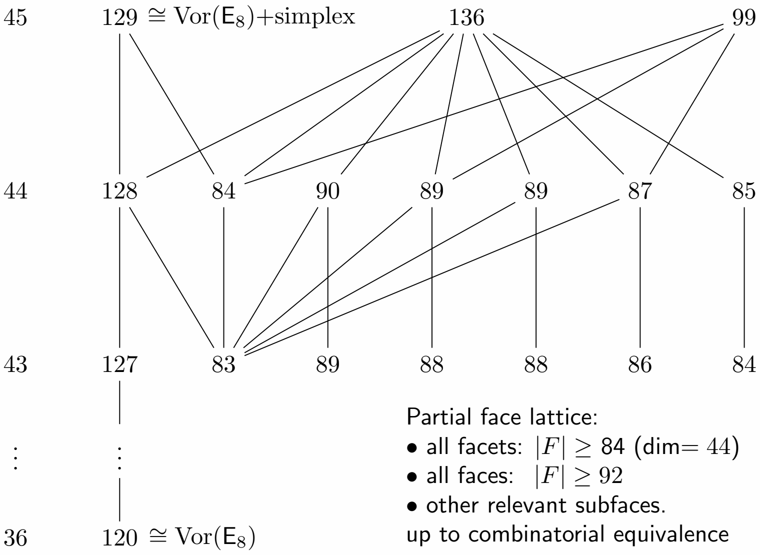
Kissing numbers

Theorem: Kissing numbers

The set of possible kissing numbers $|\mathrm{Min}\,(\boldsymbol{L})|$, for a lattice $\boldsymbol{L} \subset \mathbb{R}^{\mathbf{9}}$ of dimension $\mathbf{9}$, is $\mathbf{2} \cdot \{\mathbf{1}, \ldots, \mathbf{91}, \mathbf{99}, \mathbf{120}, \ldots, \mathbf{129}, \mathbf{136}\}$.



Kissing numbers



Partial face lattice:

- all facets: $|F| \geq$ 84 (dim$= 44$)
- all faces: $|F| \geq 92$
- other relevant subfaces.

up to combinatorial equivalence

Thank you!

Preprint:

https://arxiv.org/abs/2508.20719

Thank you!

Canonical functions - Examples

Graph Isomorphism: $X = \{$n-vertex graphs $\mathcal{G} = (V, E)\}$, $G = \mathrm{Sym}(n)$.

Well researched area. Babai: canonical function in quasi-polynomial time.

Important: Many practically efficient canonical functions and libraries.

PQF equivalence: $X = S^d_{>0}(\mathbb{Q})$, $G = \mathrm{GL}_d(\mathbb{Z})$, $Q \circ U := U^t Q U$

Difficulty: infinite size orbits. **Idea:** G also acts on finite set $\mathrm{Min}\,(Q)$

"A canonical form for positive definite matrices" [**DS**HV**vW**20]. $\longrightarrow$ **GI**

Polyhedral Cone: $X = \{\{v_1, \ldots, v_m\} \subset \mathbb{R}^n\}$, $G = \mathrm{GL}_n(\mathbb{R})$

"Computing symmetry groups of polyhedra" [B**DS**PRS14] $\longrightarrow$ **GI**

Face equivalence: $X = \{$faces of $P\}$, $G \subset \mathrm{Aut}(P)$.

Permutation group acting on sets: "Minimal and Canonical images" [JJPW19]

Face equivalence

- Each face can be described by the set of rays $F \subset [m]$ contained in it.
- Polyhedral symmetry group can be described as a permutation group $G \subset \mathrm{Sym}_m$.
- $X = \{F \subset [m] : F \text{ is a face of } \mathcal{P}\}$, $\sigma \circ F = \sigma(F) = \{\sigma(x) : x \in F\}$.
- Define total ordering $\preccurlyeq$ on $\mathcal{P}([m])$, then

$$\theta_m(F) = \min_{\preccurlyeq}(\mathrm{Orb}(G, F))$$

  is canonical. Use stabilizer chain to calculate $\theta_m(F)$ without full enumeration.
- (*Minimal and Canonical images*, JJPW, 2017): dynamical ordering tailored for each orbit. Constructed in a canonical way during the algorithm.
- Up to multiple orders of magnitude faster. (**1 min. vs 2 ms** in GAP)
- **Mathieu** ported the GAP routines and the package to C++: **even faster**.