A Short Survey of Cryptography Based on the Lattice Isomorphism Problem

Wessel van Woerden (PQShield).

PQ SHIELD

Motivation

LWE, SIS, NTRU lattices: versatile, but poor geometry.

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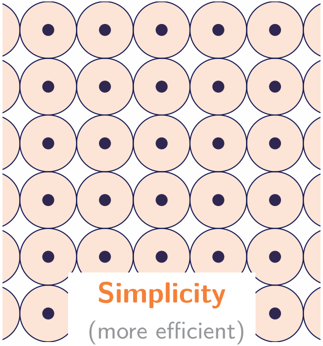
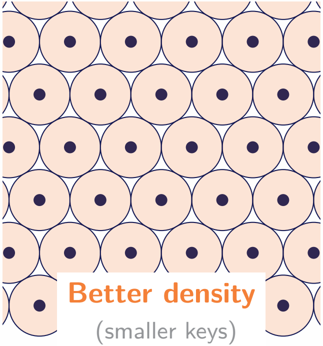
Many remarkable lattices exist with great geometric properties.





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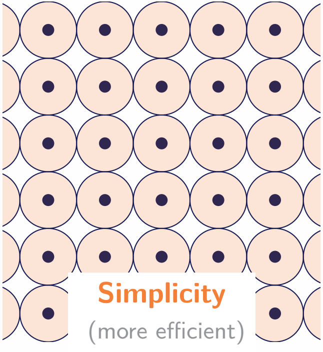

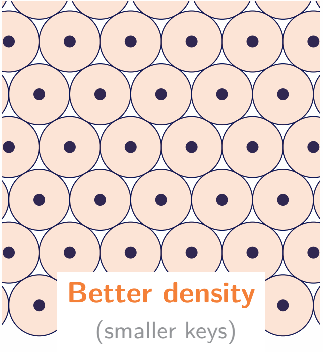

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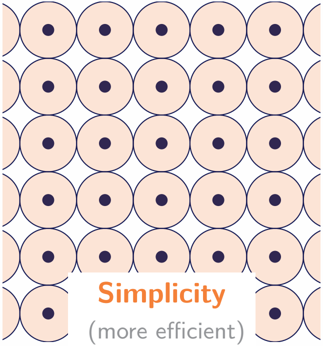
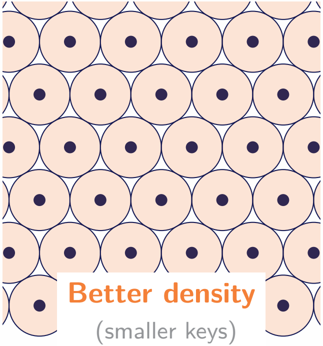


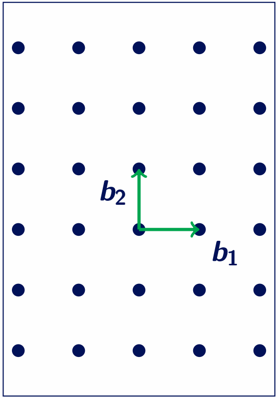


Can we use these in cryptography?

Lattice Isomorphism Problem: yes, we can!

Lattices and decoding

Lattice $\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$



integer lattice $\mathbb{Z}^n$

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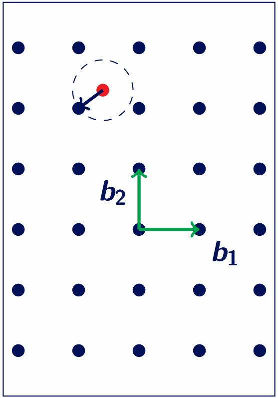

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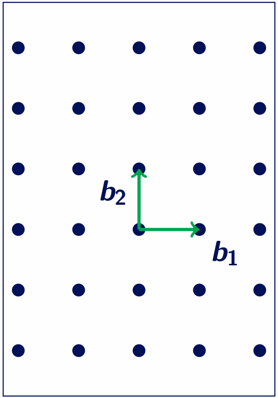

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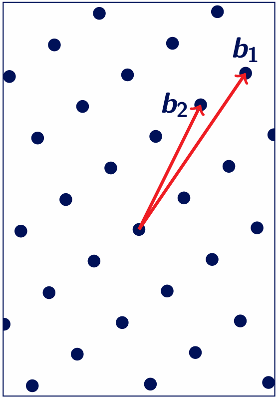

random lattice

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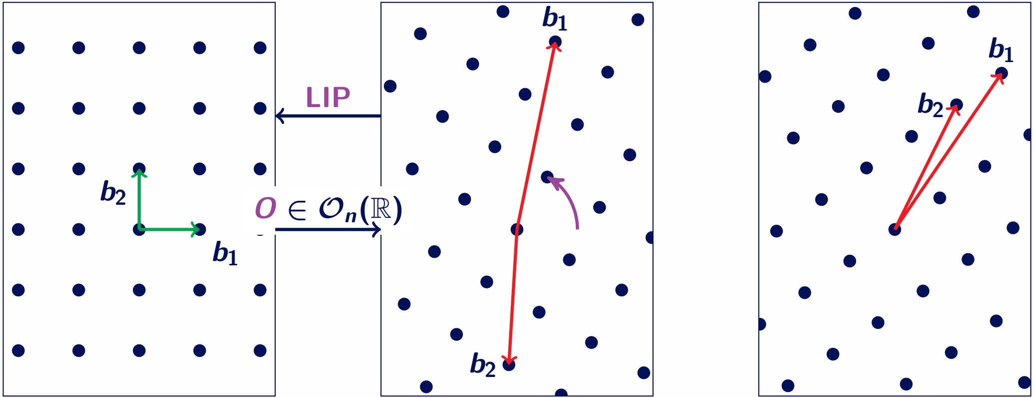

integer lattice $\mathbb{Z}^n$

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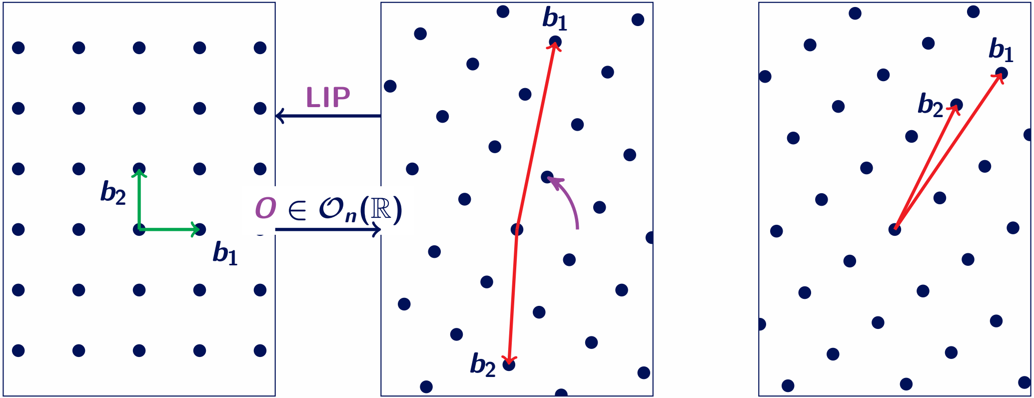

integer lattice $\mathbb{Z}^n$ — decoding easy

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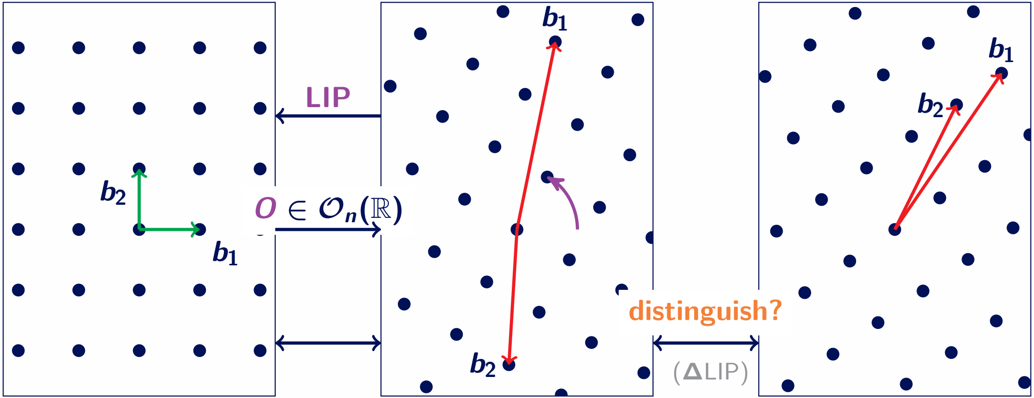

integer lattice $\mathbb{Z}^n$	$\mathbb{Z}^n$ rotated	random lattice
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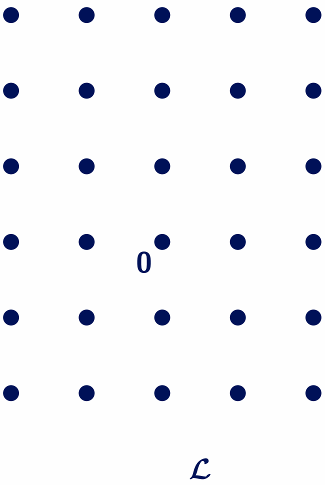
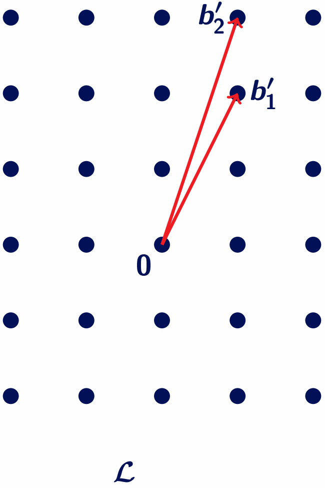
Encryption scheme based on LIP

Decodable lattice



Bad basis of $\mathcal{L}$



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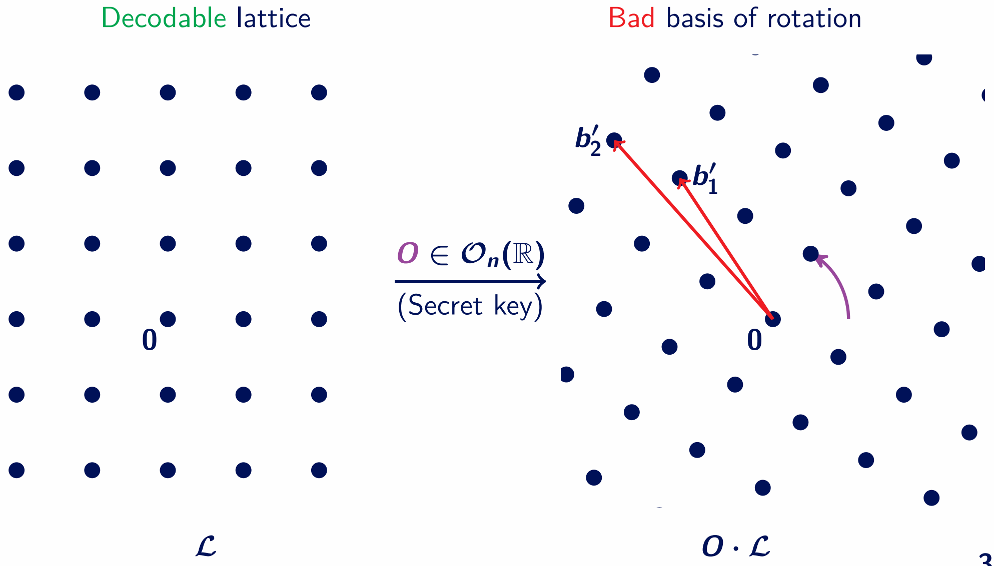

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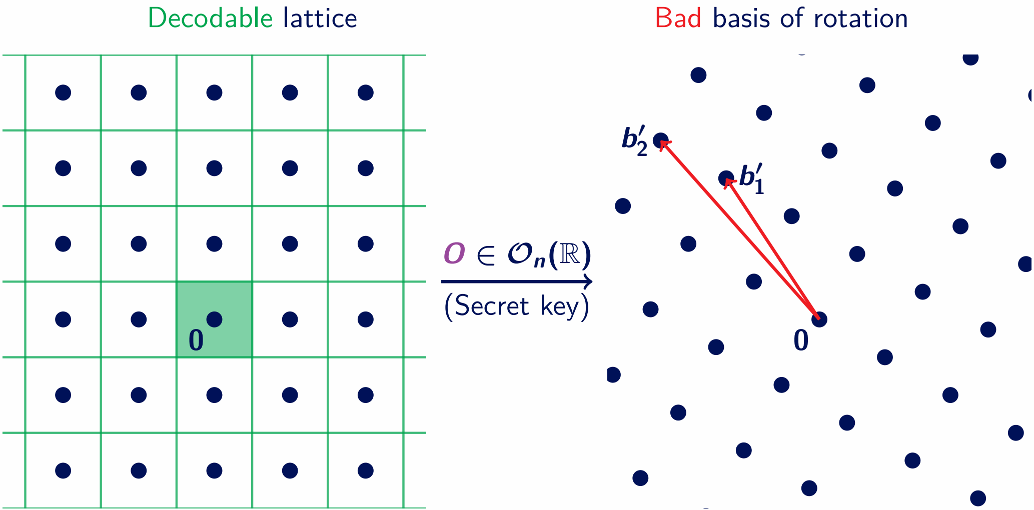

Hides (decoding) structure of $\mathcal{L}$

Encryption scheme based on LIP

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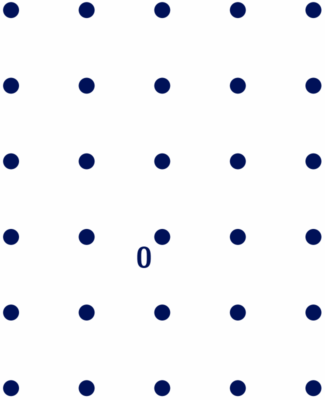
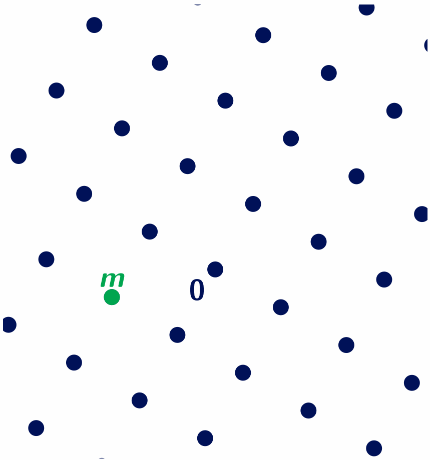


$$\xrightarrow[\text{(Secret key)}]{O \in \mathcal{O}_n(\mathbb{R})}$$

Bad basis of rotation



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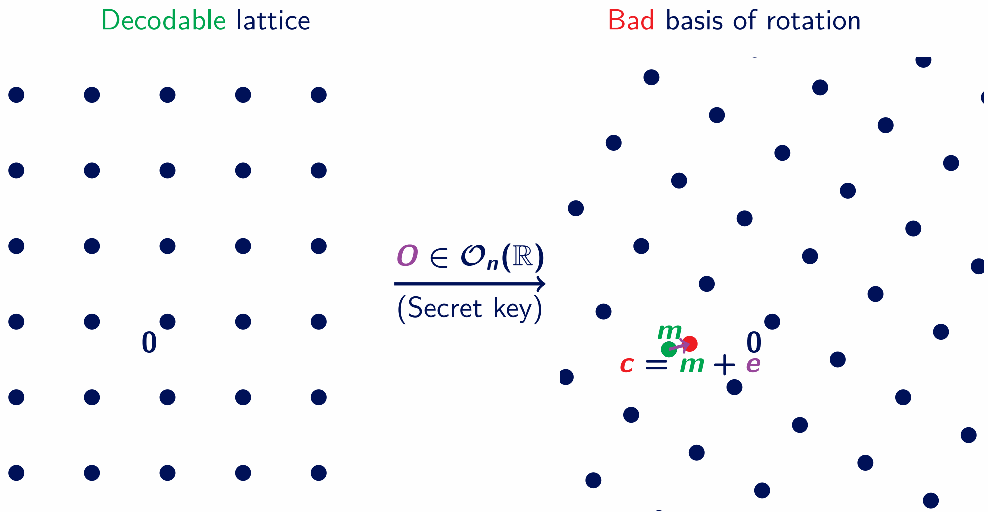
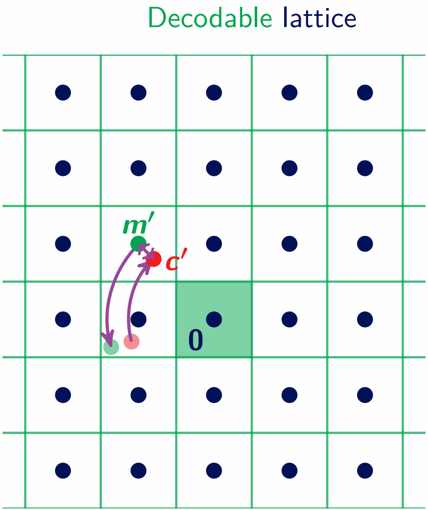


Encrypt by adding a small error

Encryption scheme based on LIP



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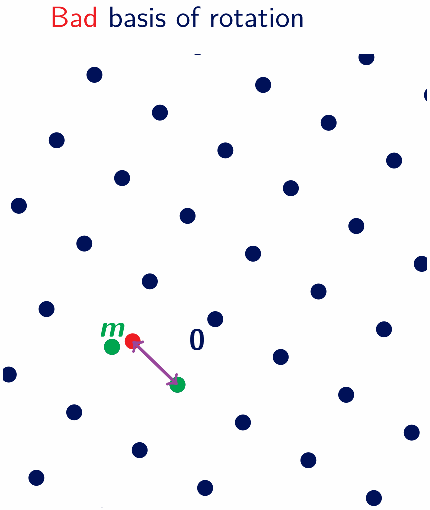

Decrypt using decoding algorithm

The beginnings

On LIP, quadratic forms, remarkable lattices, and cryptography. [D**vW**22]

Identification Scheme
LIP

KEM&PKE
ΔLIP

Signature Scheme
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Just How Hard Are Rotations of $\mathbb{Z}^n$ [BGPSD23]

$\mathbf{1}$-bit PKE based on $\mathbf{\Delta}$LIP with $\mathbb{Z}^n$.

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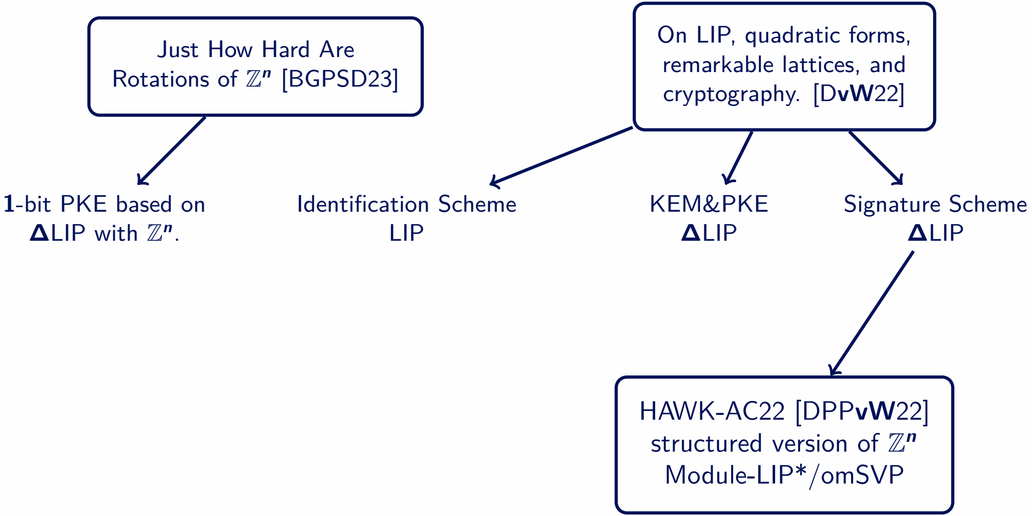

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1-bit PKE based on **Δ**LIP with $\mathbb{Z}^n$.

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ΔLIP

Signature Scheme
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HAWK-AC22 [DPP**vW**22]
structured version of $\mathbb{Z}^n$
Module-LIP*/omSVP

HAWK [HAWK23,FH23]
fast & compact & fp-free
NIST submission

Simplicity of $\mathbb{Z}^n$ + module-LIP*

$\Downarrow$

competitive signature scheme (vs FN-DSA!)

[1] On Ludo's Laptop

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Fast:	KeyGen: **3.5** ms[1]	Sign/Verify: $<$ **0.1** ms
Compact:	$\|$🔑$_{\text{pub}}\| =$ **1024** bytes	$\|\text{sig}\| =$ **555** bytes
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Only lattice scheme in round **2** of NIST call for additional signature schemes.

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HAWK - a Signature Scheme from $\mathbb{Z}^n$

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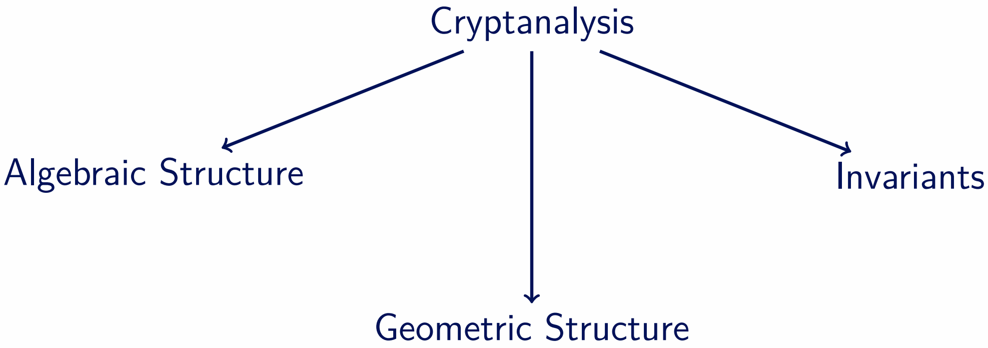
PQC forum Sept. 12:
'HAWK is a very cool scheme'

Cryptanalysis



Cryptanalysis

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Algebraic Structure

rank 2 attacks

(symplectic) automorphisms

[LJPW24,vGP25]

Geometric Structure

Invariants

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algebraic genus

special/spinor genus

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(efficiency ⟶ structure)

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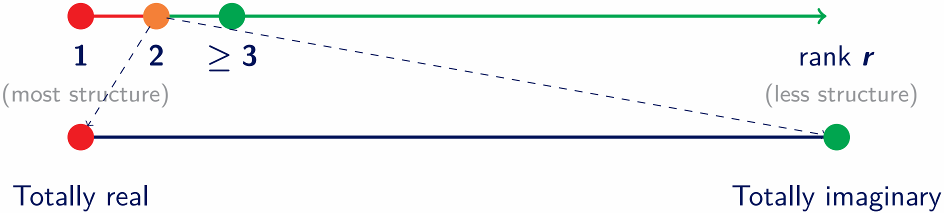

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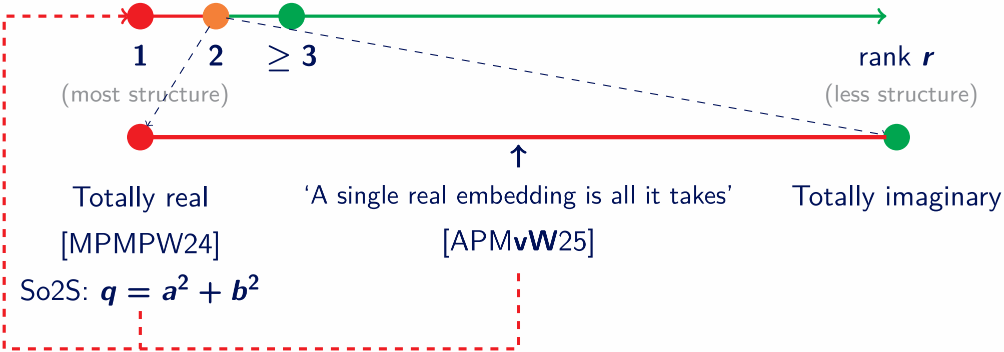

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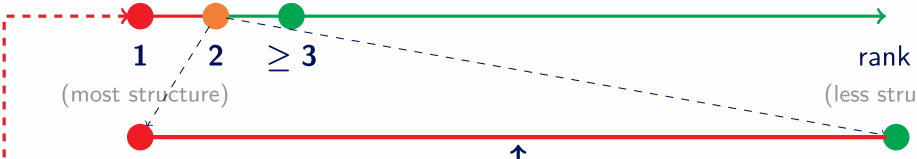


Totally real

[MPMPW24]

So2S: $q = a^2 + b^2$

'A single real embedding is all it takes'

[APM**vW**25]

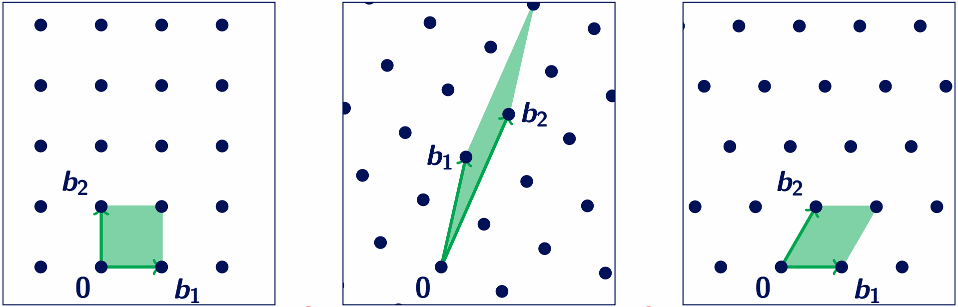
Totally imaginary

HAWK (CM field)

[CMEPMPW25,CM25]:

So4S: $q = a^2 + b^2 + c^2 + d^2$

attack does not work



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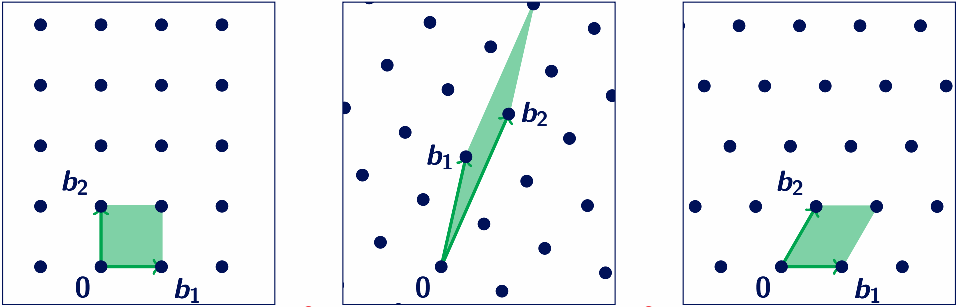
Invariants



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Lemma:

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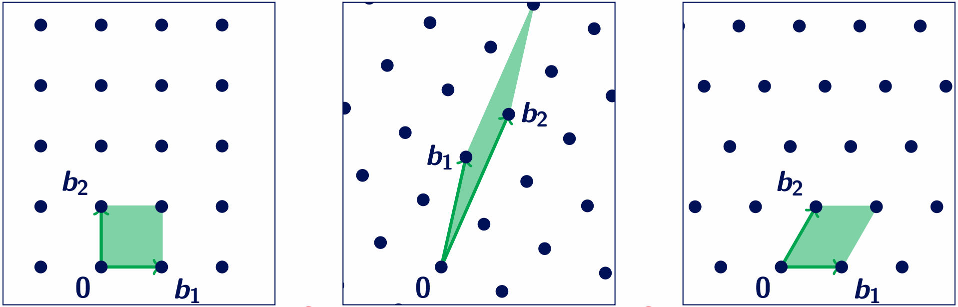

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Conjecture [DvW22]: **Genus is the strongest** efficiently computable invariant.

Genus Invariant

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[vG25] $> \mathbf{2^{1050}}$ module lattices have the same genus as HAWK.

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Conclusion: invariants do not seem to affect security.

Constructions

KEM&PKEs

$\mathbb{Z}^{\boldsymbol{n}}$ or $\mathrm{BW}_{\boldsymbol{n}} + \boldsymbol{\Delta}$LIP

[ARLW24,CBZIPC24]

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LIP (group action)

[JWLLGPW25,LJPW25]

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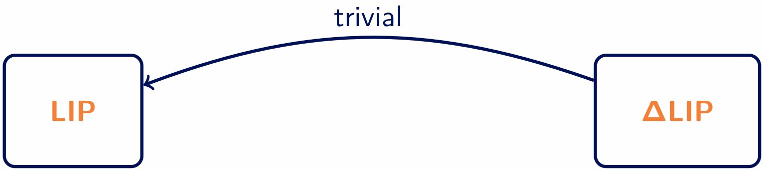
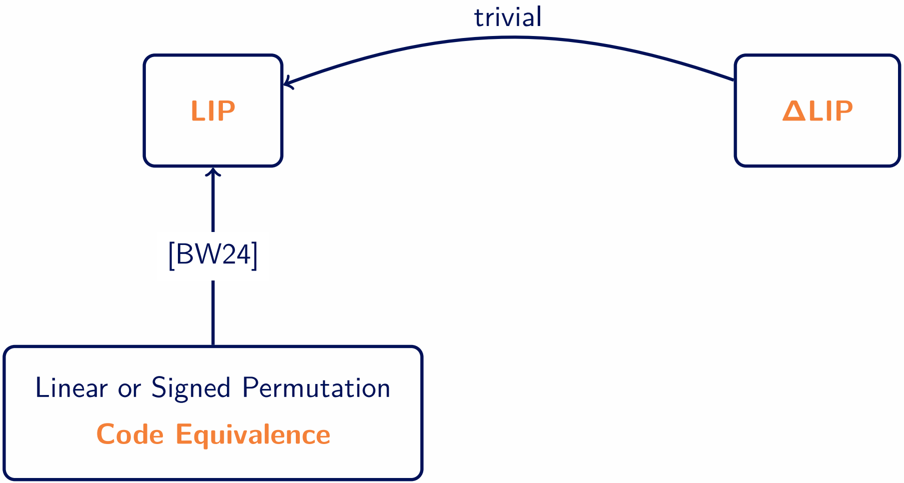
Allows for **Advanced Cryptographic Constructions**

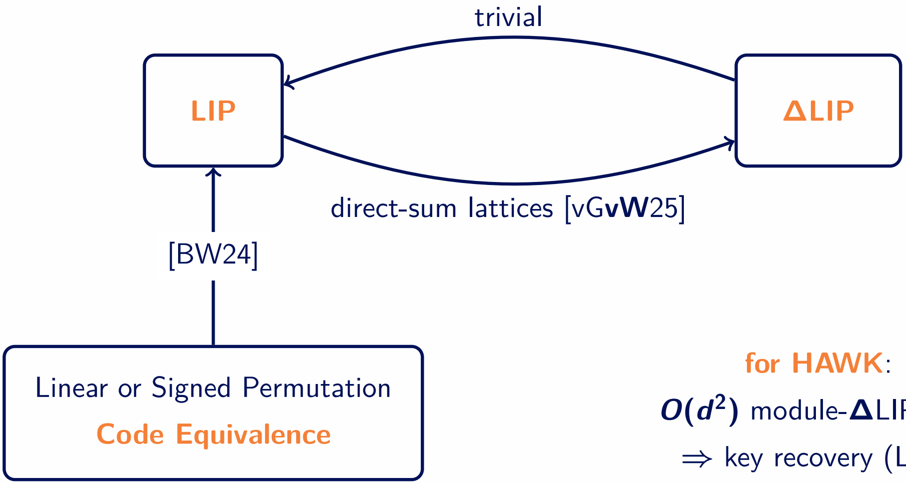
Reductions



Reductions


trivial
LIP
ΔLIP
[BW24]
Linear or Signed Permutation
Code Equivalence

Reductions



for HAWK:

$O(d^2)$ module-**Δ**LIP calls

$\Rightarrow$ key recovery (LIP).

The Future of LIP?

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Cryptanalysis: module-LIP & remarkable lattices

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Cryptanalysis: module-LIP & remarkable lattices

Thank you!

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