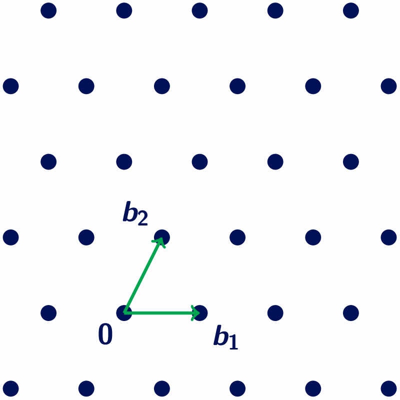
Dense and smooth lattices in any genus

Wessel van Woerden (PQShield).

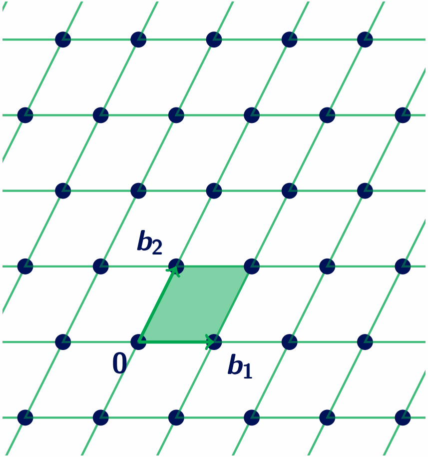


Lattice

$\mathbb{R}$-linearly independent $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$

$\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$,

basis B, gram matrix $G := B^t B$



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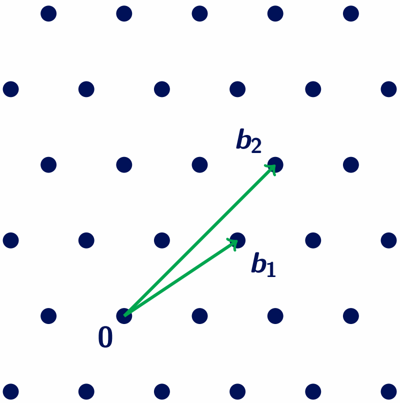
Lattice (co)volume

$$\det(\mathcal{L}) := \mathrm{vol}(\mathbb{R}^n/\mathcal{L}) = |\det(B)|$$



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Infinitely many distinct bases

$B' = B \cdot U$, $G' = U^t G U$,

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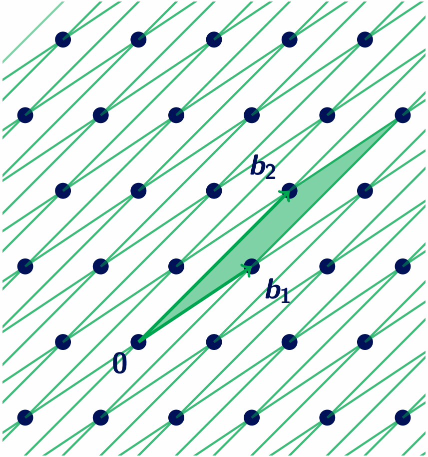

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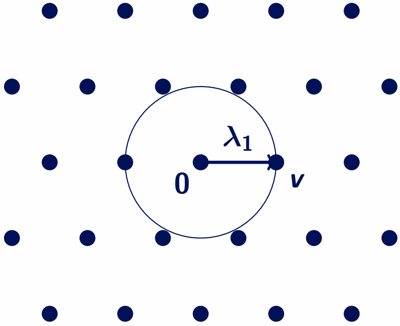
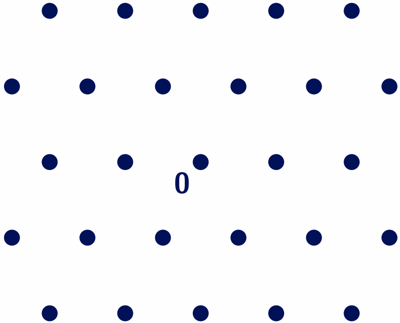


First minimum & theta series

$$\lambda_1(\mathcal{L}) := \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2$$



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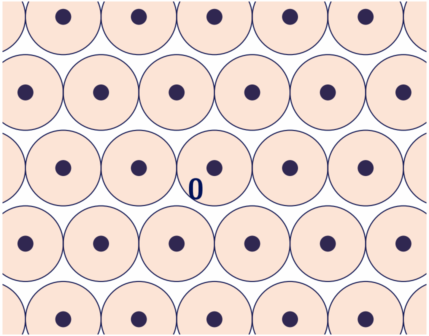
Lattice packings



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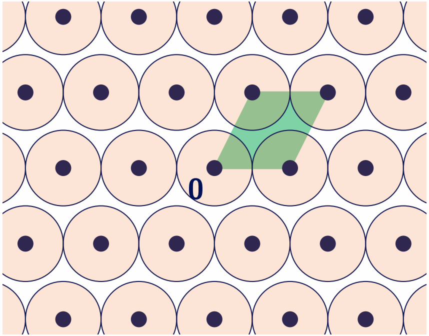
Packing density

$$\delta(\mathcal{L}) = \frac{\mathrm{vol}(\mathcal{B}_R^n)}{\det(\mathcal{L})}, \text{ where } R = \frac{1}{2}\lambda_1(\mathcal{L})$$



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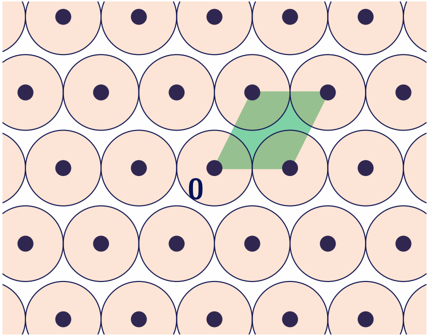

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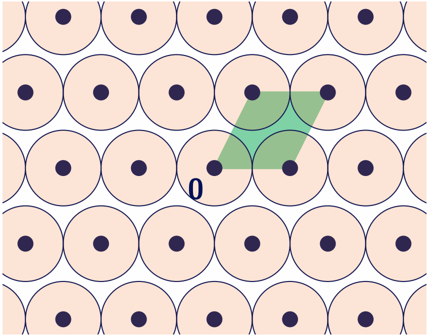
$$\lambda_1(\mathcal{L}) \leq 2 \cdot \underbrace{\frac{\det(\mathcal{L})^{1/n}}{\mathrm{vol}(\mathcal{B}_1^n)^{1/n}}}_{\mathrm{Mk}(\mathcal{L})}$$



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Minkowski-Hlawka Theorem

There exists a lattice $\mathcal{L} \subset \mathbb{R}^n$ with $\lambda_1(\mathcal{L}) > \mathrm{gh}(\mathcal{L}) := \frac{1}{2}\mathrm{Mk}(\mathcal{L})$.

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Definition (Siegel 1945): Haar measure

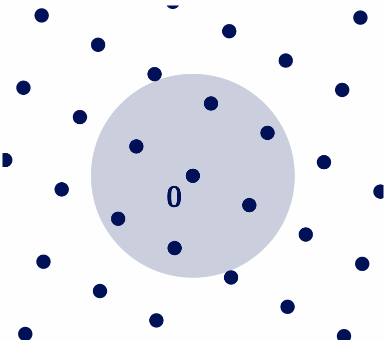
The Haar measure on $\mathcal{SL}_n(\mathbb{R})$ has finite mass on the quotient space of unit volume lattices $\mathcal{L}_{[n]} = \mathcal{SL}_n(\mathbb{R})/\mathcal{SL}_n(\mathbb{Z})$.

Average number of lattice points: Hlawka43, Siegel45

Let $\mathcal{L}_{[n]}$ be the space all lattices of dimension n and volume 1, then

$$\mathop{\mathbb{E}}_{\mathcal{L}\in\mathcal{L}_{[n]}} |\mathcal{L} \cap \lambda \cdot \mathcal{B}^n| = 1 + \mathrm{vol}(\lambda \cdot \mathcal{B}^n).$$

'Average of one non-zero point per unit volume'



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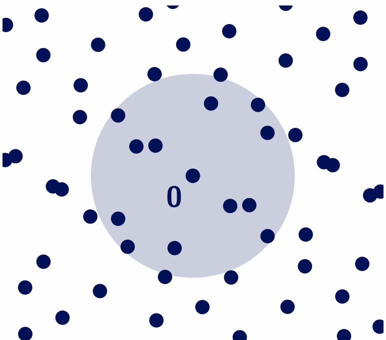

Averaging formula and the Minkowski-Hlawka Theorem

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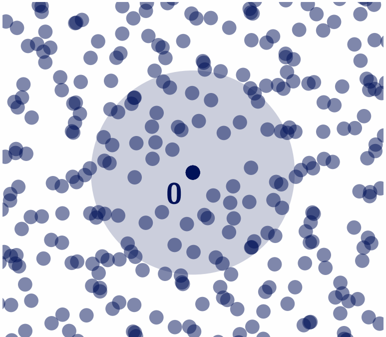

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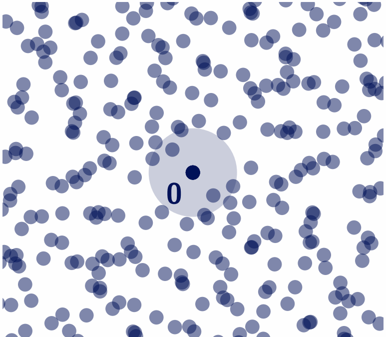

Proof: Minkowski-Hlawka Theorem

Pick $\lambda = \frac{1}{2}\,\mathrm{Mk}(n)$,

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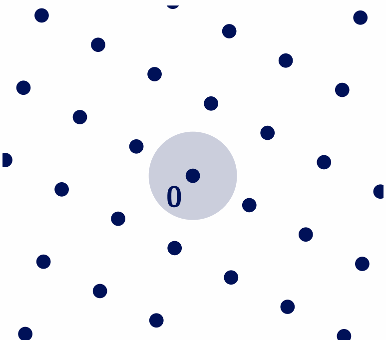

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$\Rightarrow \exists \mathcal{L} \in \mathcal{L}_{[n]}$ with $|\mathcal{L} \cap \lambda \cdot \mathcal{B}^n| \leq 2$,

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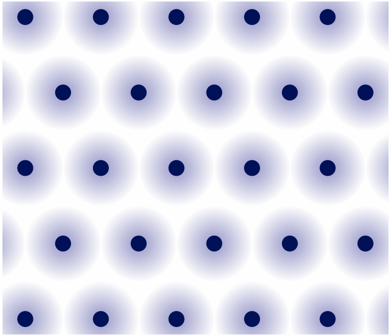

Smoothing parameter

'minimum $s > 0$ such that centered Gaussian with width s is ϵ-close to uniform over $\mathbb{R}^n/\mathcal{L}$'

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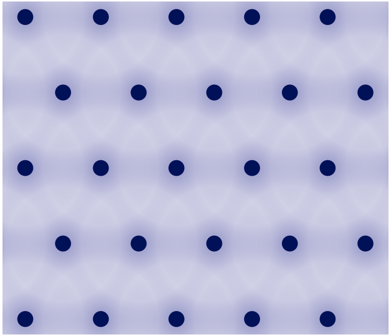

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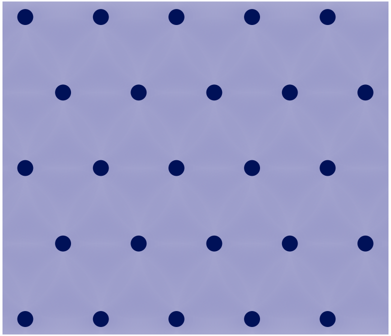

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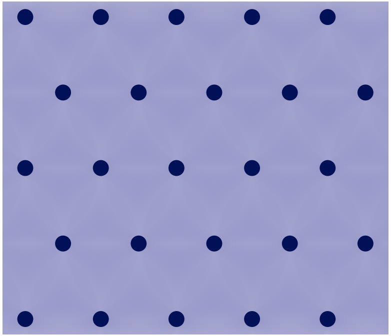

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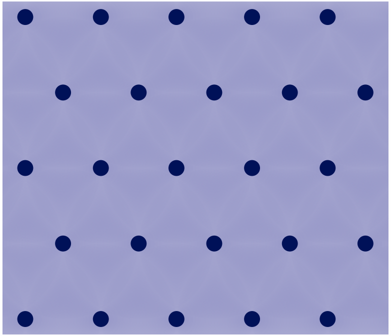
Dual lattice

$$\mathcal{L}^* := \{y \in \mathbb{R}^n : \forall x \in \mathcal{L}, \langle x, y\rangle \in \mathbb{Z}\}$$

$$\eta_{2^{-n}}(\mathcal{L}) \leq \sqrt{n}/\lambda_1(\mathcal{L}^*)$$



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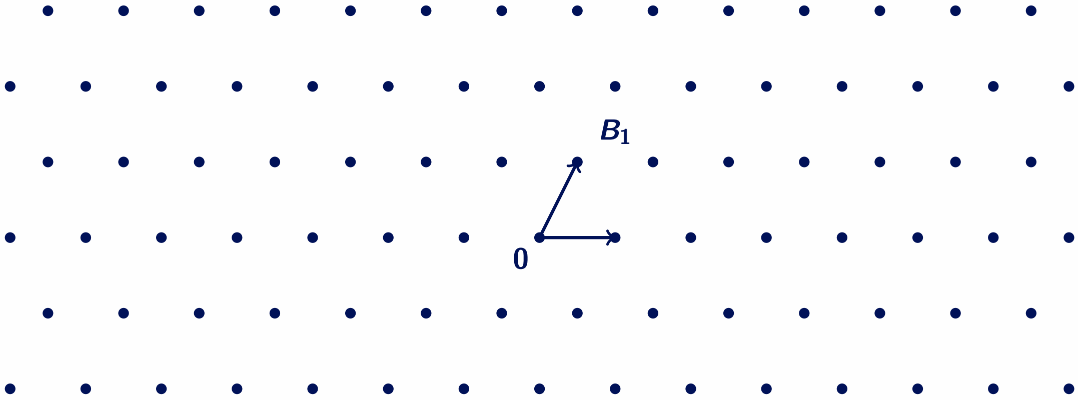
Good smoothing: $\epsilon \in (e^{-n}, 1]$

For a random lattice $\mathcal{L}^*$, $\theta_{\mathcal{L}^*}(\exp(-\pi s^2)) \leq 1 + s^{-n}\det(\mathcal{L})$

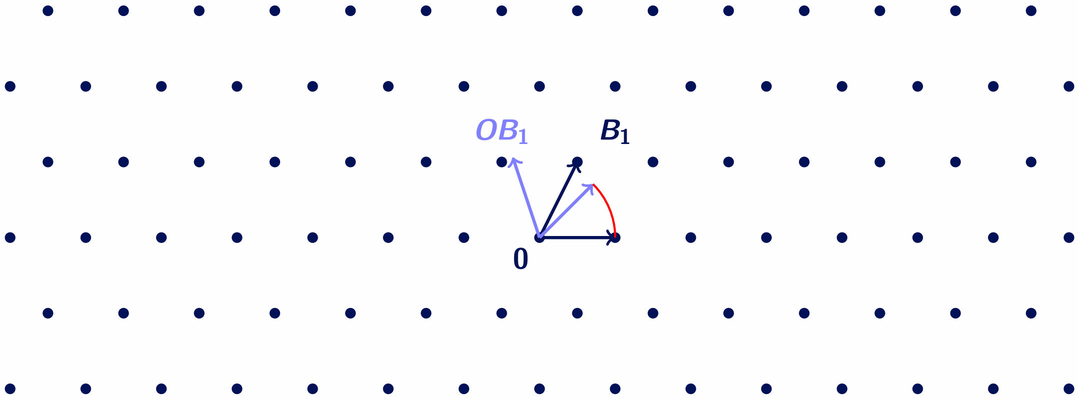
$\Rightarrow$ there exists a lattice with $\eta_\varepsilon(\mathcal{L}) \leq (\det(\mathcal{L})/\varepsilon)^{1/n}$.

LIP and the genus of a lattice



$\mathcal{L}_1 = \mathcal{L}(B_1)$

Lattice Isomorphism Problem (LIP)



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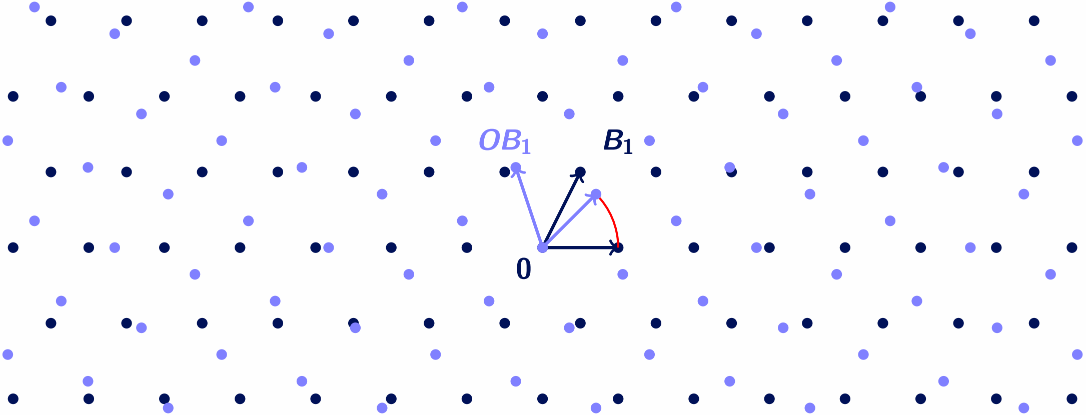

$\mathcal{L}_1 = \mathcal{L}(B_1)$ $\mathcal{L}_2 = O \cdot \mathcal{L}(B_1)$

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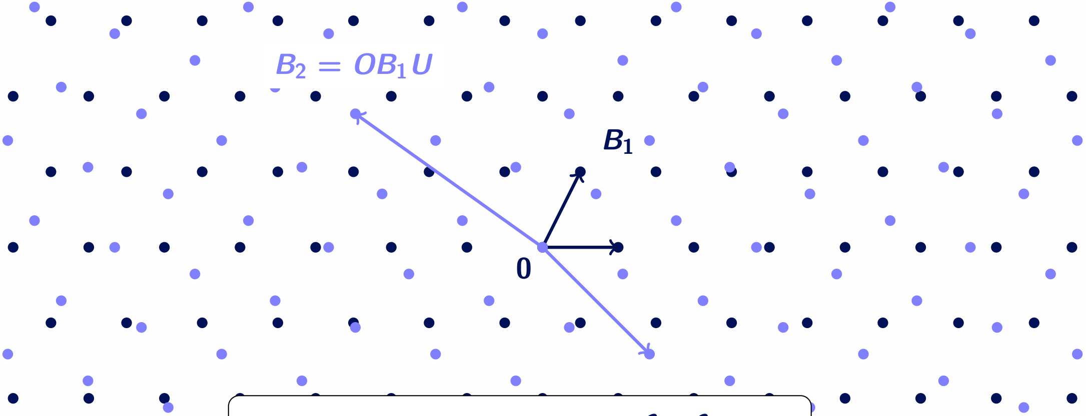

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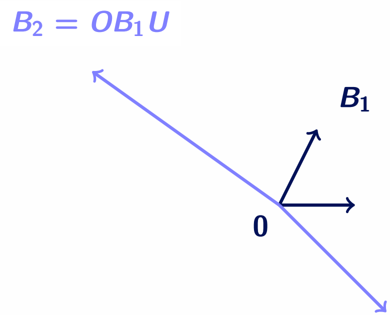

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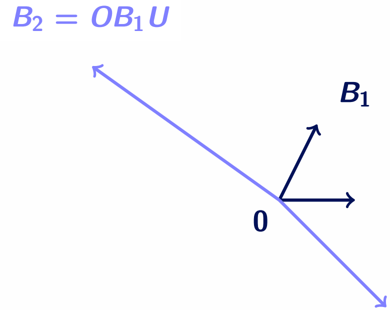

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$\mathcal{L}_2 = O \cdot \mathcal{L}(B_1)$

(unique up to $\mathrm{Aut}(\mathcal{L}) := \{O \in \mathcal{O}_n(\mathbb{R}) : O \cdot \mathcal{L} = \mathcal{L}\}$)

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$$\mathcal{L}(B_1) \cong \mathcal{L}(B_2)$$

$$\Longleftrightarrow$$

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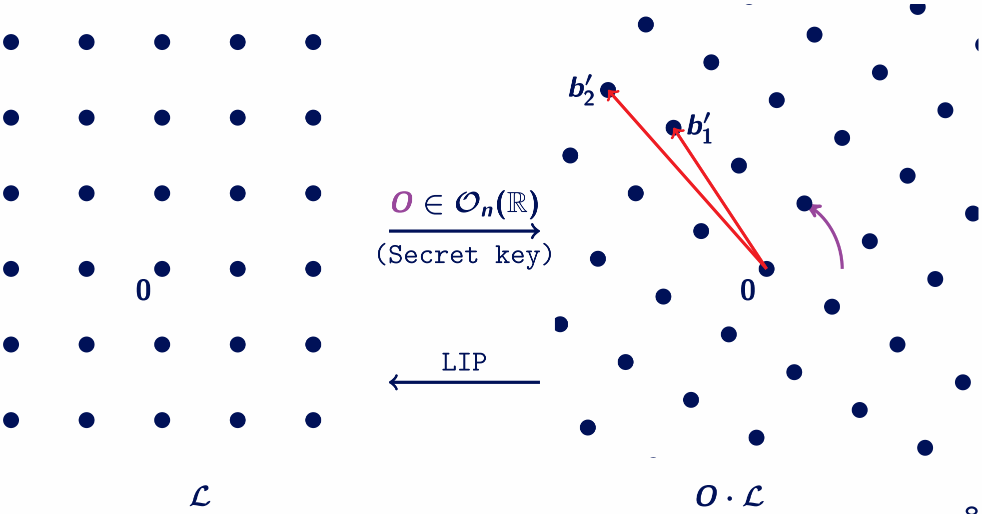
- We restrict to integer gram matrices $G := B^t B$.

Encryption scheme from LIP (informal)

Decodable lattice   Bad basis of rotation



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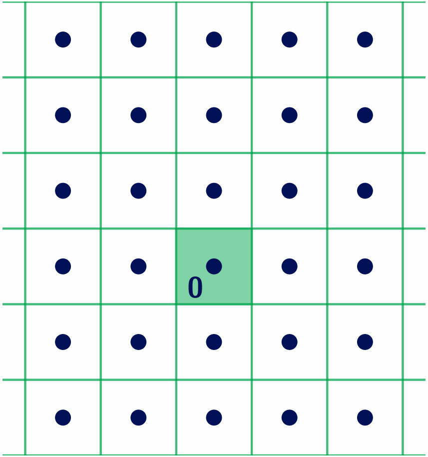
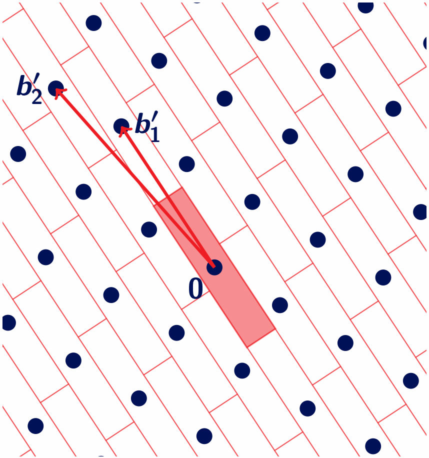


$\boldsymbol{O} \in \mathcal{O}_n(\mathbb{R})$

(Secret key)

Bad basis of rotation



Hides (decoding) structure of $\mathcal{L}$

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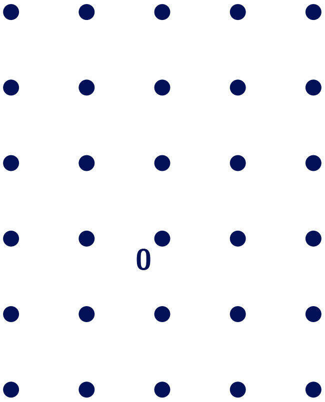
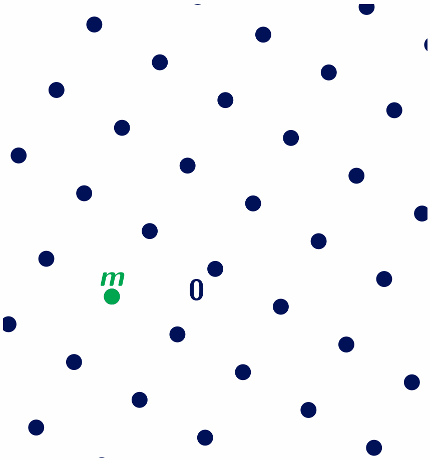
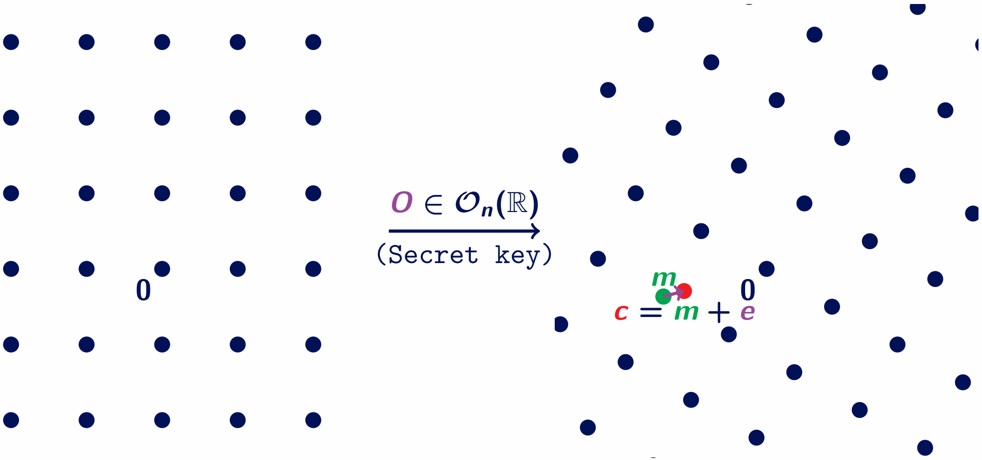


$$\xrightarrow[\text{(Secret key)}]{O \in \mathcal{O}_n(\mathbb{R})}$$



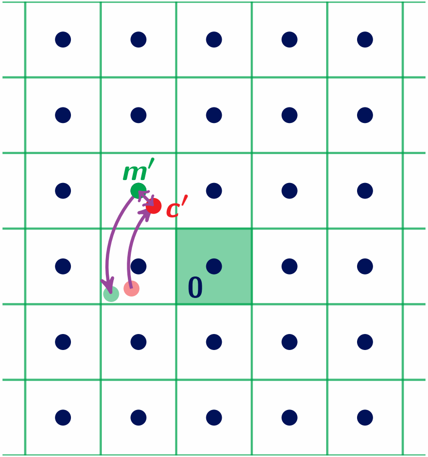
Encryption scheme from LIP (informal)

Decodable lattice | Bad basis of rotation



Encrypt by adding a small error

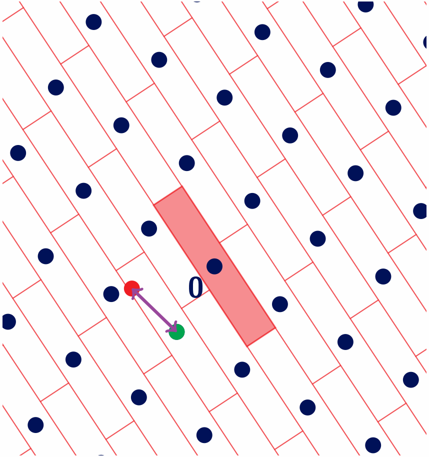
Decodable lattice



$O \in \mathcal{O}_n(\mathbb{R})$

(Secret key)

Bad basis of rotation



Decrypt using decoding algorithm

Cryptography from LIP

- LIP as a new hardness assumption

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DvW, EC 2022: On LIP, QFs, Remarkable Lattices, and Cryptography

Use LIP to hide a remarkable lattice:

- Identification, Encryption and Signature scheme

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DPPvW, AC 2022: HAWK scheme

Efficient signature scheme based on module-LIP on $\mathbb{Z}^n$

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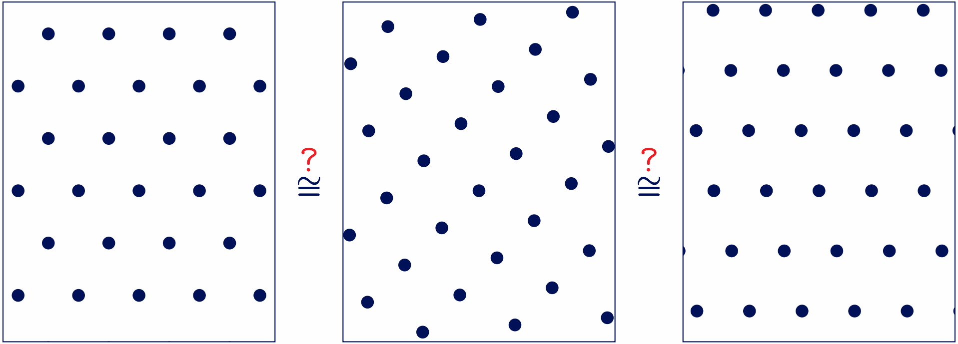
- Many other works using LIP appeared recently

Distinguish LIP

Definition: distinguish LIP (Δ-LIP)

Let $\mathcal{L}_1, \mathcal{L}_2$ be two non-isomorphic lattices and let $b \leftarrow \{1, 2\}$ uniform.
Given $\mathcal{L} \in [\mathcal{L}_b]$, recover b.



$\mathcal{L}_1$ $\quad$ $O \cdot \mathcal{L}_b$ $\quad$ $\mathcal{L}_2$

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Usual security assumption:

Given:

1. some remarkable lattice $\mathcal{L}_1$
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Goal: find an auxiliary lattice with the right geometric properties

Example: good packing, smoothing, covering..

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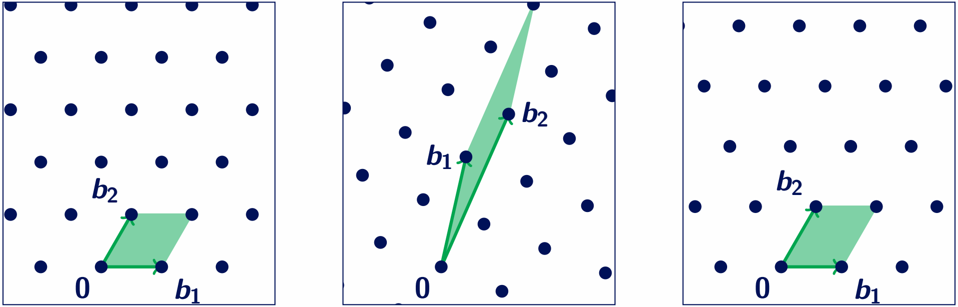
(Search to distinguish reduction for direct sum lattices: eprint 2025/1204)

Invariants



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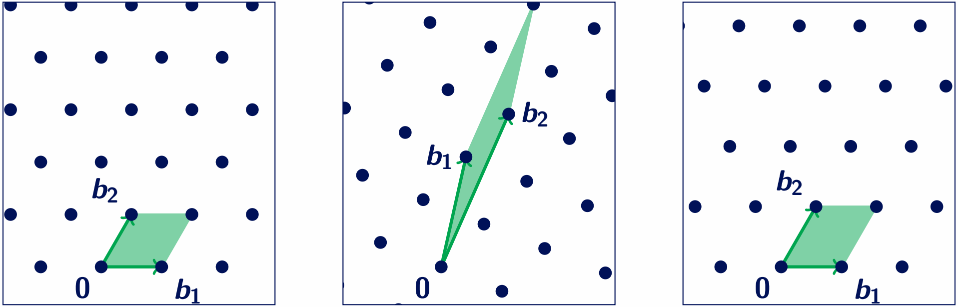

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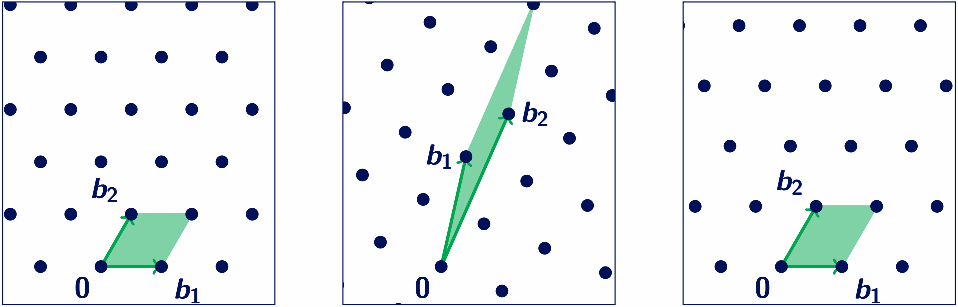

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$\Rightarrow$ auxilary lattice must have same (polytime-computable) invariants

- We consider integral lattices: $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in \mathcal{L}$

Genus:

Two integral lattices $\mathcal{L}_1, \mathcal{L}_2 \subset \mathbb{R}^n$ are in the same genus if

$$\mathcal{L}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{L}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{for all primes } p,$$

where $\mathbb{Z}_p$ are the p-adic integers. ($\Leftrightarrow U^t G_1 U = G_2$ for some $U \in \mathcal{GL}_n(\mathbb{Z}_p)$)

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How restricting is the genus invariant?

Dense lattices in any genus

BGPSD, EC 2023: Just how hard are rotations of $\mathbb{Z}^n$?

Do there exist lattices $\mathcal{L} \in \mathbf{gen}(\mathbb{Z}^n)$ with

- $\lambda_1(\mathcal{L}) \geq \Omega(\mathsf{Mk}(\mathcal{L})/\sqrt{\log(n)})$, or
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- Need: a Minkowski-Hlawka-like Theorem within any fixed genus

Mass formula and the size of a genus

Theorem: Smith-Minkowski-Siegel mass formula (Siegel, 1935)

Any genus $\mathcal{G}$ contains a finite number of isom. classes and its mass

$$M(\mathcal{G}) := \sum_{[\mathcal{L}]\in\mathcal{G}} \frac{1}{|\mathrm{Aut}(\mathcal{L})|},$$

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- Question: do these behave like random lattices?

Random distribution over genus

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- Comes with similar average point counting results!

  ($\Longrightarrow$ Minkowski-Hlawka like theorem?)

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- Two integral lattices $\mathcal{L}_1, \mathcal{L}_2$ are p-neighbours $\mathcal{L}_1 \sim_p \mathcal{L}_2$ if

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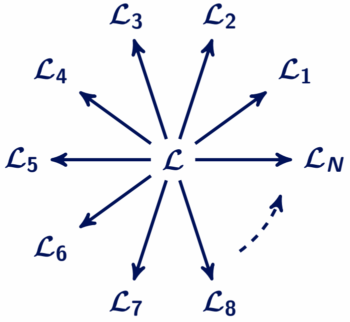

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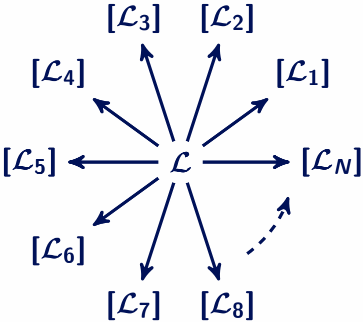

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$$[\mathcal{L}_1] \;\underset{\mathcal{L}_1 \sim_p \mathcal{L}_2}{\overline{\qquad\qquad\qquad}}\; [\mathcal{L}_2]$$

- Random walk: $\mathcal{L}_1 \sim_p \mathcal{L}_2 \sim_p \ldots \sim_p \mathcal{L}_k$ where $\mathcal{L}_{i+1}$ is a uniformly random p-neighbour of $\mathcal{L}_i$.

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- Random walk: $\mathcal{L}_1 \sim_p \mathcal{L}_2 \sim_p \ldots \sim_p \mathcal{L}_k$ where $\mathcal{L}_{i+1}$ is a uniformly random p-neighbour of $\mathcal{L}_i$.
- For large enough p, a random walk has limit distribution $\mathcal{D}(\mathcal{G})$.
  $\implies$ efficient sampling algorithm for $\mathcal{D}(\mathcal{G})$.

Results (1) - Good (dual) packing

Theorem (good packing): Minkowski-Hlawka theorem for fixed genus

Let $\mathcal{G}$ be any genus of dimension $n \geq 6$ such that $\mathrm{rk}_{\mathbb{F}_p}(\mathcal{G}) \geq 6$ for all primes p. Let $C = \frac{7\zeta(3)}{9\zeta(2)} \approx 0.57$. Then there exists a $\mathcal{L} \in \mathcal{G}$ with

$$\lambda_1(\mathcal{L})^2 \geq \left\lceil (C \cdot \det(\mathcal{L})/\omega_n)^{2/n} \right\rceil \approx n/2\pi e \cdot \det(\mathcal{L})^{2/n} = \mathrm{gh}(\mathcal{L})^2.$$

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- Essentially matches packing density of a random lattice.
- Similar result for simultaneous good primal and dual packing.
- For a constant $0 < c \leq 1$ we get that

$$\mathbb{P}\left[\lambda_1(\mathcal{L}) \geq \left\lceil c^2 \cdot (C \cdot \det(\mathcal{L})/\omega_n)^{2/n} \right\rceil\right] > 1 - c^n.$$

Theorem (good smoothing):

'...' Let $C = 26.1$ and $\varepsilon \in [e^{-n}, 1)$, then there exists a lattice $\mathcal{L} \in \mathcal{G}$ with $\eta_\varepsilon(\mathcal{L}^*) \leq (C \cdot \det(\mathcal{L}^*)/\varepsilon)^{\frac{1}{n}}$.

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- Corollary: $\exists \mathcal{L}^* \in \mathcal{G}^*$ with a good covering radius

The hammer: Siegel-Weil mass formula

Theorem: Siegel-Weil mass formula - average point counting

For any genus $\mathcal{G}$ and integer $m > 0$, the expectation

$$N_m := \mathop{\mathbb{E}}_{[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})} \left|\{x \in \mathcal{L} : \|x\|^2 = m\}\right|,$$

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Let $\mathcal{G}$ be any genus such that $\mathrm{rk}_{\mathbb{F}_p}(\mathcal{G}) \geq 6$ for all primes p. Then

$$N_m \leq \frac{9\zeta(2)}{7\zeta(3)} n \operatorname{vol}(\mathcal{B}_1^n) \cdot m^{n/2-1} / \det(\mathcal{G}) \quad \text{for all } m > 0.$$

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- Sufficient to prove main results with MH-like argument

Example: even unimodular case (1)

Definition: even unimodular lattices

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Lemma: mass formula

For $n = 8k \geq 8$, B_i the i-th Bernoulli number, and $\sigma_z(m) = \sum_{d|m} d^z$ is the sum of positive divisors function, we have

$$\Theta_{\mathcal{G}_{8k,e}}(q) = E_{4k}(q^2) = 1 + \frac{-8k}{B_{4k}} \sum_{m=1}^{\infty} \sigma_{4k-1}(m) q^{2m}.$$

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- $\Theta_{\mathcal{G}_{128,e}}(q) \approx 1 + 6.11 \cdot 10^{-37} q^2 + 5.64 \cdot 10^{-18} q^4 + 7.00 \cdot 10^{-7} q^6 + 52.01 q^8 + 6.63 \cdot 10^7 q^{10} + O(q^{12})$

- Idea: recall that:

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Lemma: existence of good packing

Let $\mathcal{G}$ be a genus with average theta series $\Theta_{\mathcal{G}}(q) = 1+\sum_{m=1}^{\infty} N_m q^m$.
If $\sum_{m=1}^{\lambda} N_m < 2$, then there exists a lattice $\mathcal{L}\in\mathcal{G}$ s.t. $\lambda_1(\mathcal{L})^2 > \lambda$.

Example: even unimodular case (3)

Lemma: even packing (Milnor, Serre, 73)

Let $n = 8k \geq 8$ with $k \in \mathbb{N}$, then there exists an n-dimensional even unimodular lattice $\mathcal{L}$ with $\lambda_1(\mathcal{L})^2 \geq 2 \cdot \left\lceil \frac{1}{2}(\frac{3}{5}\omega_n)^{-2/n} \right\rceil \approx n/2\pi e$.

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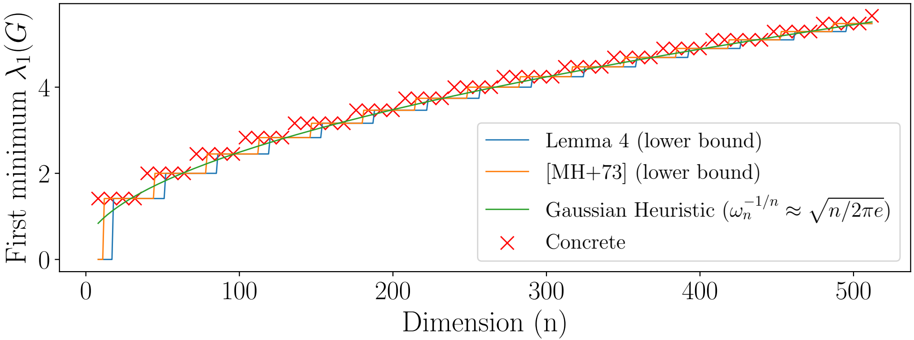

General case: compute mass formula

- We want to count the average number of solutions N_m to $f(x) := x^t G_{\mathcal{L}} x = m$ with $x \in \mathbb{Z}^n$ when $[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})$.

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Theorem: Siegel-Weil mass formula

For any genus $\mathcal{G}$ of dimension ≥ 2 and average theta series $\Theta_{\mathcal{G}}(q) = 1 + \sum_{y=1}^{\infty} N_m q^m$ we have

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- Only primes $p | 2m \det(\mathcal{G})^2$ have to be considered
- Can even be generalized to matrix equations!

  (mass formula from $M(\mathcal{G})$ follows from equation $U^t G U = G$)

- Need: upper bound on expected number N_m of solutions.

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Bounding densities

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Local density over reals

- For $f(x) = x^t x = x_1^2 + \ldots + x_n^2 \in \mathbb{R}$ the density of solutions for $f(x) = y$ is essentially the volume of the sphere of radius $\sqrt{y}$.

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- Behaves as expected:

  sparser lattice $\Rightarrow$ larger $\det(\mathcal{G})$ $\Rightarrow$ smaller coefficients.

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Let $\mathcal{G}$ be a genus with $\mathrm{rk}_{\mathbb{F}_p}(\mathcal{G}) \geq 6$ for all primes p, then for all $m \geq 0$

$$\prod_{p=2,3,\ldots} \delta_{\mathcal{G},p}(m) \leq \frac{18\zeta(2)}{7\zeta(3)} < 3.52$$

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Rough idea:

$G \sim_{\mathbb{Z}_p} G_0 + p \cdot G_1 + p^2 \cdot G_2 + \ldots$ with $\det(G_i) \neq 0 \bmod p$, $\dim(G_0) \geq 6$

If $\delta_{G_0,p}(m) \leq c$ for all $m \geq 0$, then $\delta_{G,p}(m) \leq c$ for all $m \geq 0$.

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If $\delta_{G_0,p}(m) \leq c$ for all $m \geq 0$, then $\delta_{G,p}(m) \leq c$ for all $m \geq 0$.

- Bound $\delta_{G_0,p}(m)$ using classification of p-adic normal forms.

Theorem:

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- Tight up to a constant factor
- Sufficient to prove the main results
- Conjecture: remove conditions $\Longrightarrow$ extra factor $\mathbf{poly}(y)$

  (but rather tedious to work out)

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Can we construct stronger efficiently computable invariants?

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What about module lattices? (e.g. Hermitian forms over CM fields)

WC-AC reductions:

- the random case $[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})$ is heuristically the hardest.
- from any class $[\mathcal{L}] \in \mathcal{G}$ we can efficiently step to a random class.

Can we make a worst-case to average-case reduction within a genus?

Example: SVP, SIVP, LIP

Conclusion

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Thanks!

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