The lattice packing problem in dimension 9 by Voronoi’s algorithm

Mathieu Dutour Sikirić & **Wessel van Woerden** (PQShield).

Sphere Packing Problem

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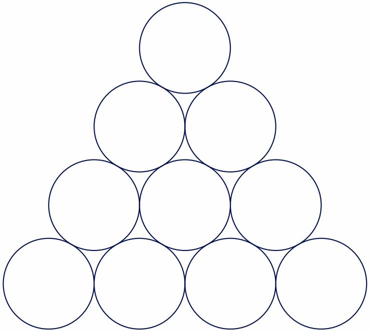

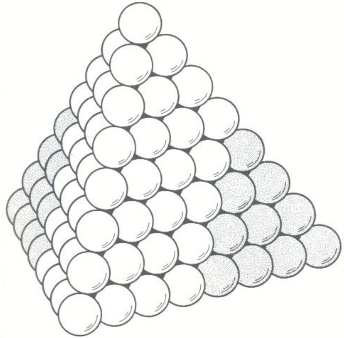

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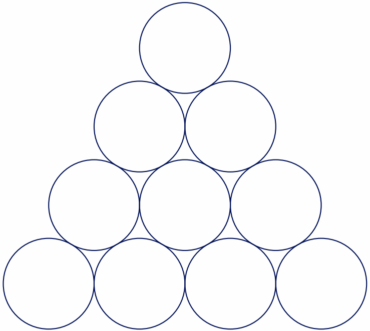

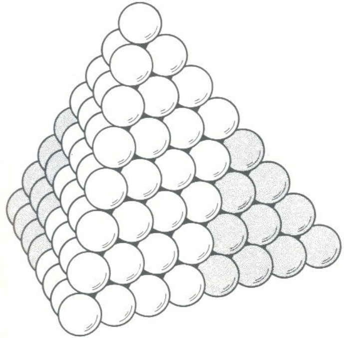

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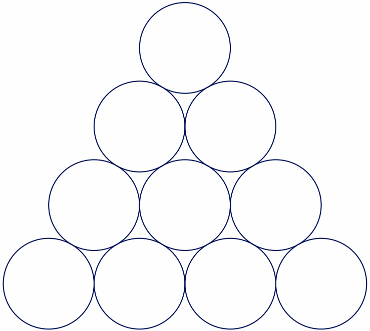

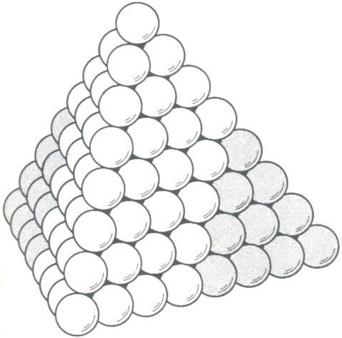

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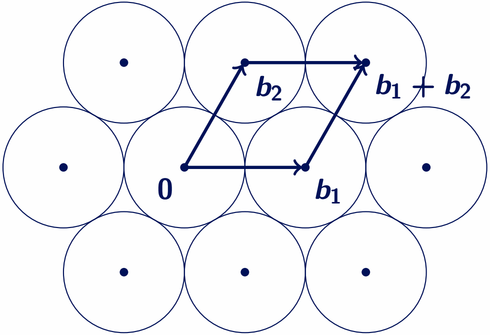
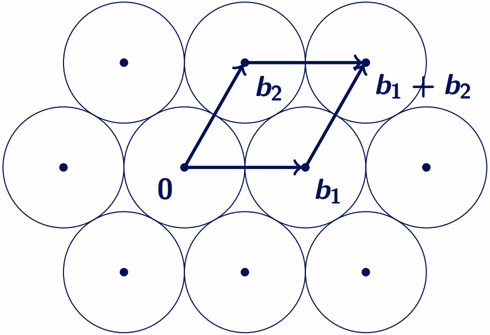
- Dimension **3** only in **1998** by a computational proof (Thomas Hales)

Lattice Packing Problem





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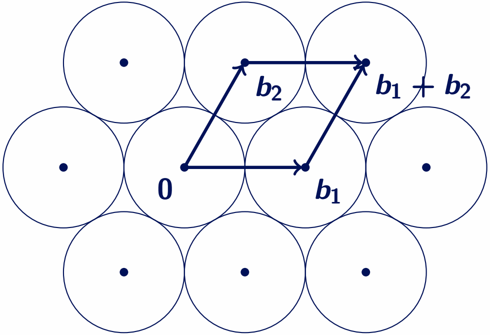

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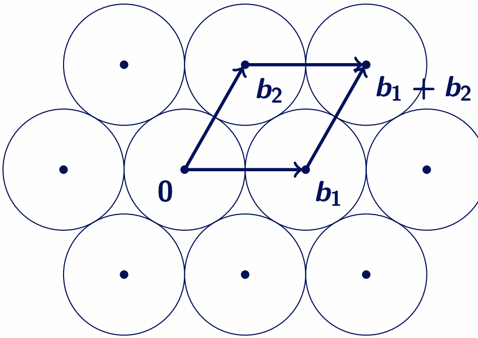

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- What about dimension 9?

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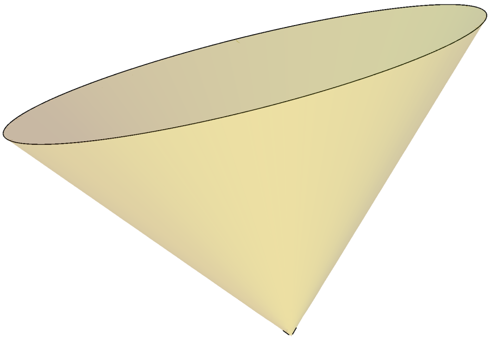
- Corollary: the laminated lattice $\boldsymbol{\Lambda_9}$ is the unique densest lattice packing.

Solution space



- Cone of positive definite matrices

$$\mathcal{S}^d_{<0} \subset \mathcal{S}^d \subset \mathbb{R}^{d\times d}.$$

$$\dim(\mathcal{S}^d) = \tfrac{1}{2}d(d+1) =: n$$

- inner product: (to show these pictures)

$$\langle A, B\rangle := \mathrm{Tr}(A^t B) = \sum_{i,j} A_{ij}B_{ij}$$

- $Q \in \mathcal{S}^d$ defines a quadratic form by

$$Q[x] := x^t Qx = \langle Q, xx^t\rangle \ \ \forall x \in \mathbb{R}^d$$

Hermite Constant

- Lattice $\boldsymbol{L} = \boldsymbol{B}\mathbb{Z}^{\boldsymbol{d}} \implies$ PQF $\boldsymbol{Q} = \boldsymbol{B}^{\boldsymbol{t}}\boldsymbol{B} \in \boldsymbol{S}^{\boldsymbol{d}}_{>\mathbf{0}}$.

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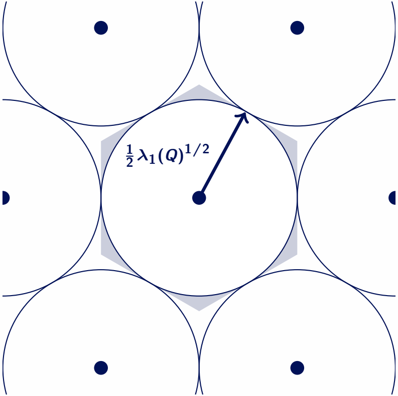
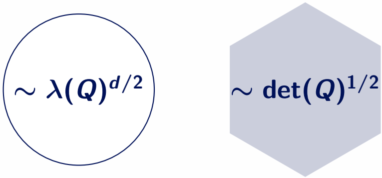


$$\lambda(\boldsymbol{Q}) := \min_{x\in\mathbb{Z}^d\setminus\{0\}} \boldsymbol{Q}[x] = \min_{y\in \boldsymbol{L}\setminus\{0\}} \|y\|^2$$

$$\operatorname{Min} \boldsymbol{Q} := \{x \in \mathbb{Z}^d : \boldsymbol{Q}[x] = \lambda(\boldsymbol{Q})\}$$



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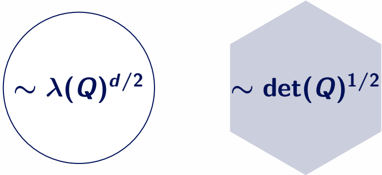

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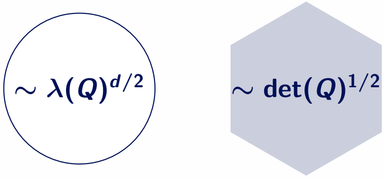

- Hermite invariant:

$$\gamma(\boldsymbol{Q}) = \frac{\lambda(\boldsymbol{Q})}{(\det \boldsymbol{Q})^{1/d}} \sim \text{density}(\boldsymbol{L})^{2/d}$$

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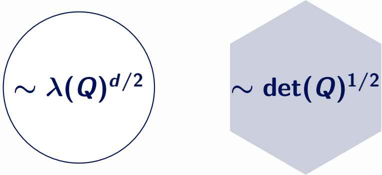


- Hermite invariant:

$$\gamma(Q) = \frac{\lambda(Q)}{(\det Q)^{1/d}} \sim \text{density}(L)^{2/d}$$

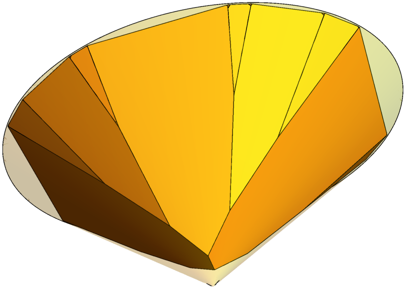
- Lattice packing problem $\Leftrightarrow$ determine Hermite's constant:

$$\gamma_d := \sup_{Q \in \mathcal{S}^d_{>0}} \gamma(Q)$$

Ryshkov Polyhedra

- For $\lambda > 0$ we define the Ryshkov Polyhedra

$$\mathcal{P}_\lambda = \{Q \in \mathcal{S}^d_{>0} : \lambda(Q) \geq \lambda\}$$



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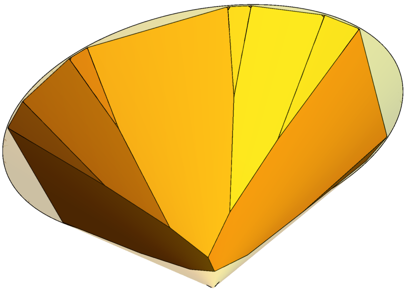

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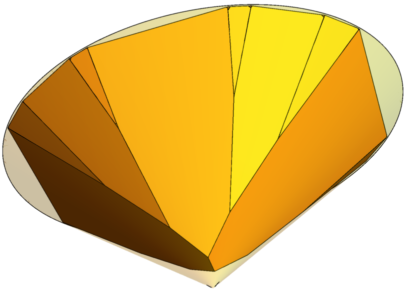

- Each facet corresponds to some primitive $\pm x \in \mathbb{Z}^d$.
- Locally finite

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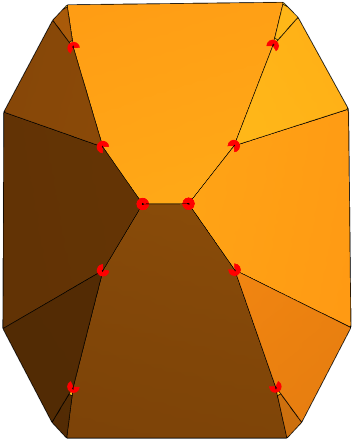
- Minkowski: $\det(Q)^{1/d}$ is (strictly) concave on $\mathcal{S}^d_{>0}$

$\Longrightarrow$ Local optima at vertices of $\mathcal{P}_\lambda$. (uses that $\mathcal{P}_\lambda$ is locally finite)

Perfect forms



- $\boldsymbol{Q}$ is perfect $\Leftrightarrow$ $\boldsymbol{Q}$ is a vertex of $\mathcal{P}_{\boldsymbol{\lambda(Q)}}$.

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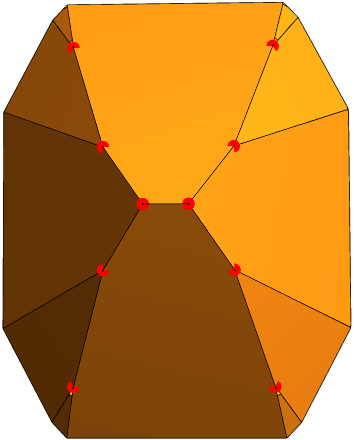

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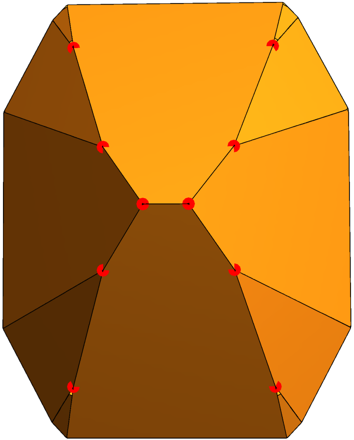

- Q is perfect $\Leftrightarrow$ Q is a vertex of $\mathcal{P}_{\lambda(Q)}$.

  $\Leftrightarrow$ Q is fully determined by Min Q and $\lambda_1(Q)$.
- Facets adjacent to $Q \leftrightarrow \pm x \in$ Min Q

  $\implies |\text{Min } Q| \geq 2n = d(d+1)$
- Voronoi's algorithm: enumerate all perfect forms (up to equivalence/similarity)

Equivalence and similarity

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Theorem: Voronoi (1908)

Up to similarity there are only a finite number of perfect forms in each dimension

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In practice..

d	# p_d
2	1 (Lagrange, 1773)
3	1 (Gauss, 1840)
4	2 (Korkine & Zolotarev, 1877)
5	3 (Korkine & Zolotarev, 1877)
6	7 (Barnes, 1957)
7	33 (Jaquet, 1993)
8	10916 (DSV, 2005)
9	$\geq$ 500.000 (DSV, 2005)
	$\geq$ 23.000.000 (**vW**, 2018)

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	Many more, to be continued...

Voronoi's Algorithm
Challenges & Solutions

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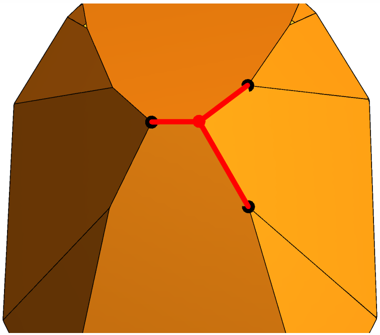

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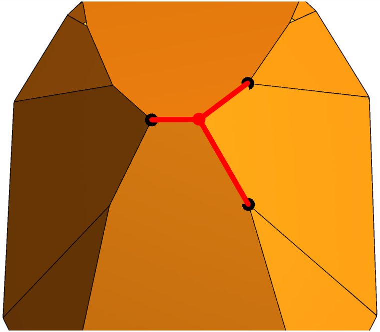

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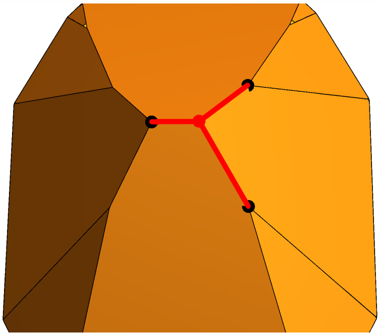

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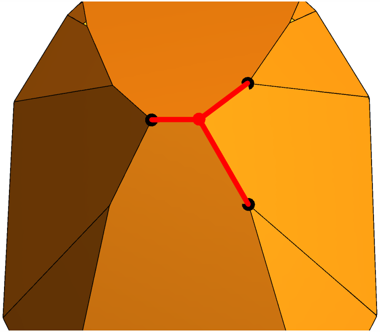

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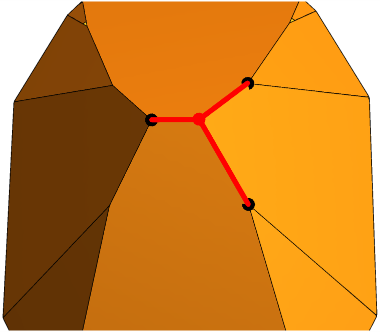

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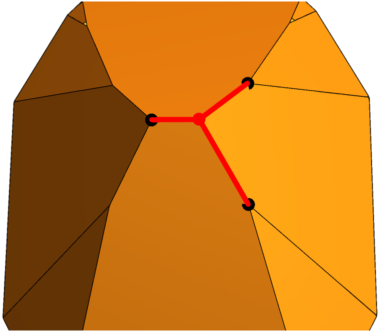

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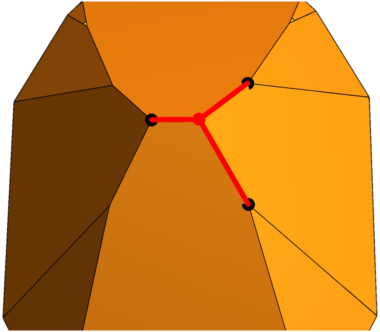

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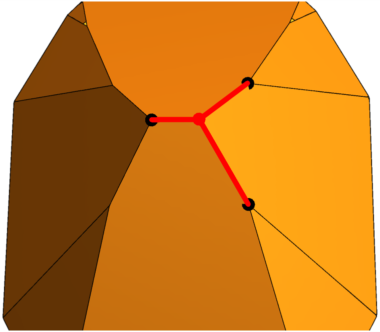

Canonical functions

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- Used for: PQF's, permutation groups acting on sets, polyhedra.

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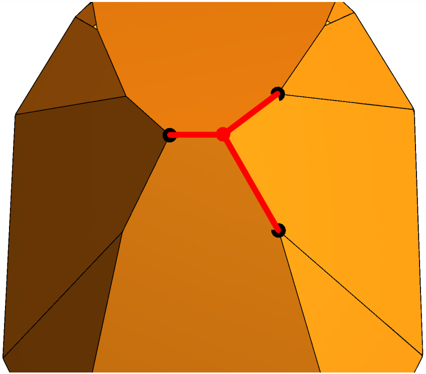
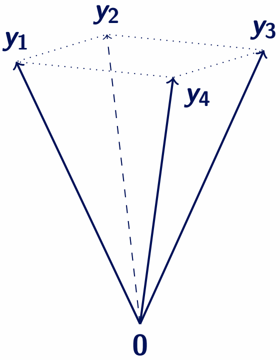
- Details: A canonical form for positive definite matrices. [ANTS 2020, DHV**vW**]

Dual Description Problem

- A (pointed) polyhedral cone $\mathcal{C} \subset \mathbb{R}^n$ can either be given by facet inequalities or by extreme rays.





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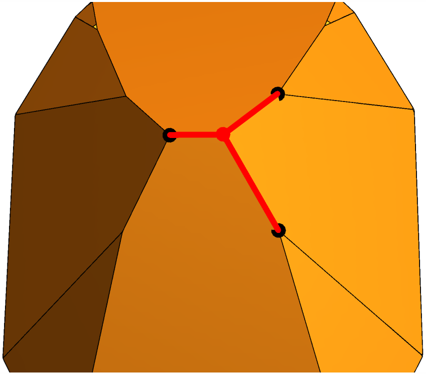

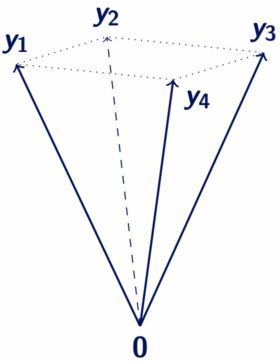

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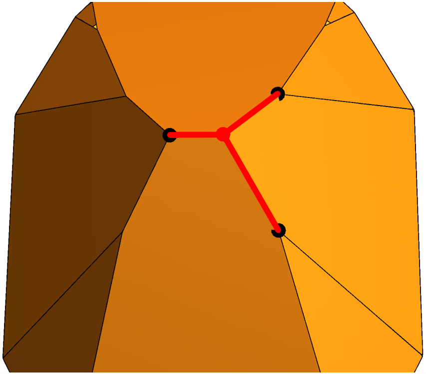

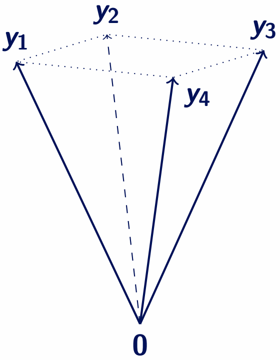

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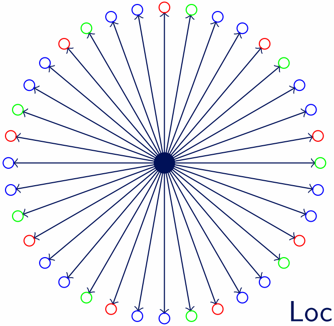
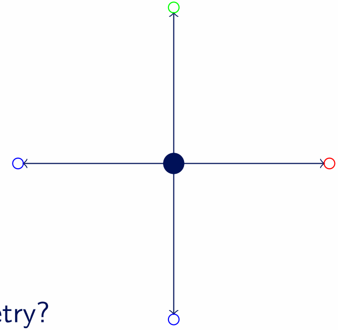
- Many rays point to equivalent forms: $\boldsymbol{Q} + \alpha_1 \boldsymbol{R_1} \sim \boldsymbol{Q} + \alpha_2 \boldsymbol{R_2}$



Local symmetry?



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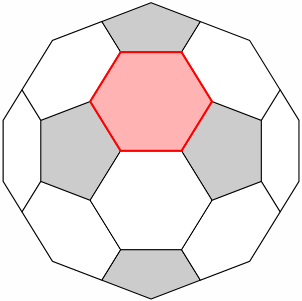
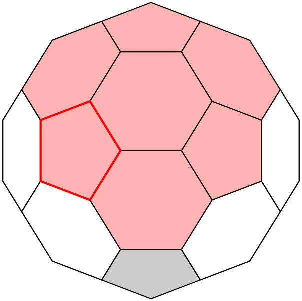
- Even harder: $\mathcal{P}(\boldsymbol{Q}_{\boldsymbol{\Lambda}_9})$ has **136** facets in a **45**-dimensional space.

Adjacency Decomposition Method



$\Rightarrow$



- Find an initial facet.
- For each new facet:
  - Determine neighbouring facets. ($\longleftarrow$ DD dimension lower!)
  - Add neighbour if orbit not already found. ($\longleftarrow$ orbit equality check)

Adjacency Decomposition Method

- Best explained in dual setting:

$$\mathcal{C} = \mathrm{cone}(\{y_1, \ldots, y_m\} \subset \mathbb{R}^n$$

  with $G \subset \mathrm{Aut}(\mathcal{C})$.

1. Find at least one facet F.
2. Determine facets $H_1, \ldots, H_k$ of F, .i.e. ridges of F in $\mathcal{C}$.
3. For all i compute facet F_i of $\mathcal{C}$ such that $H_i = F \cap F_i$.
   Keep F_i if G-inequivalent to all found facets.
4. Repeat **(2)** and **(3)** for each new facet.

- Step **(2)** is again Dual Description problem but dimension $n - 1$ and only with extreme rays contained in F.
- If still large degeneracy, recurse: $G' = \mathrm{Stab}(G, F)$.

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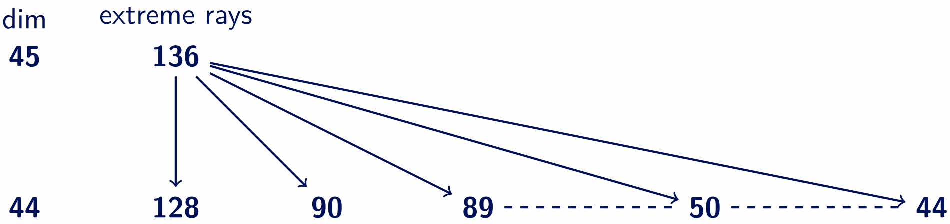
- **Mathieu** ported the GAP routines and the package to C++: even faster.

Recursive Adjacency Decomposition Method



Recursive Adjacency Decomposition Method

dim	extreme rays				
45	**136**				
44	**128**	**90**	**89**	**50**	**44**

Found facets
(canonical image)

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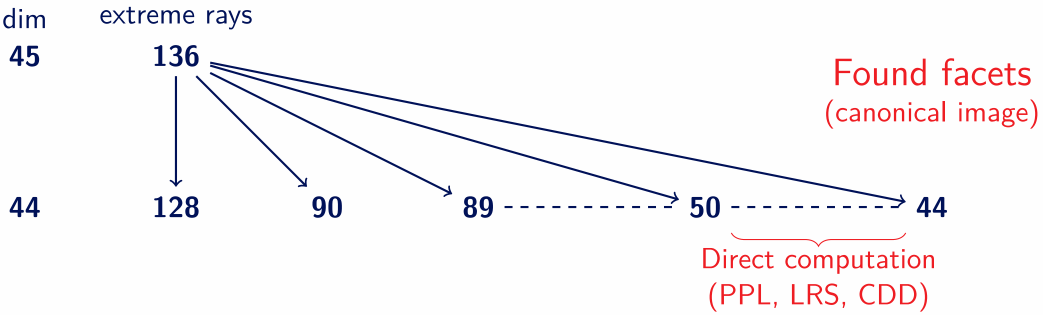

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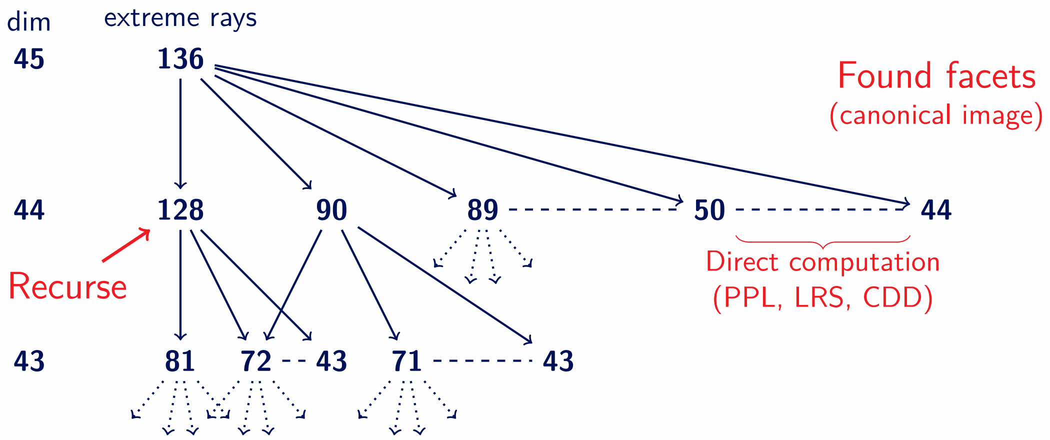

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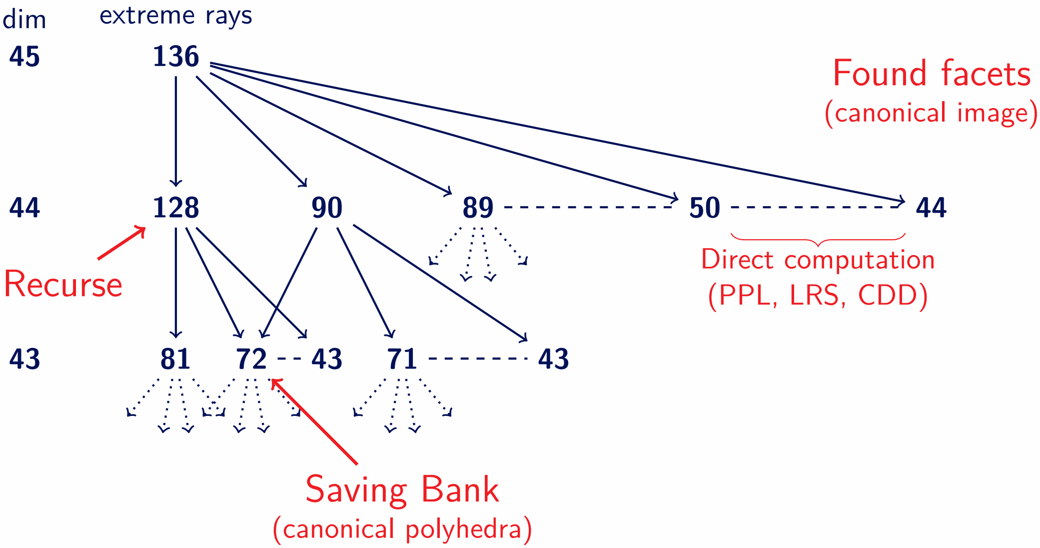
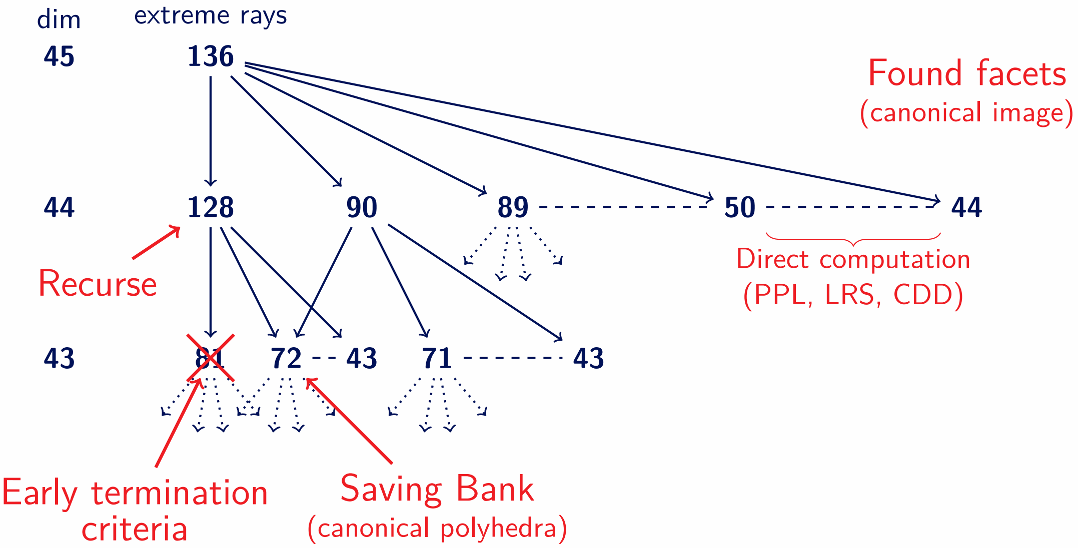
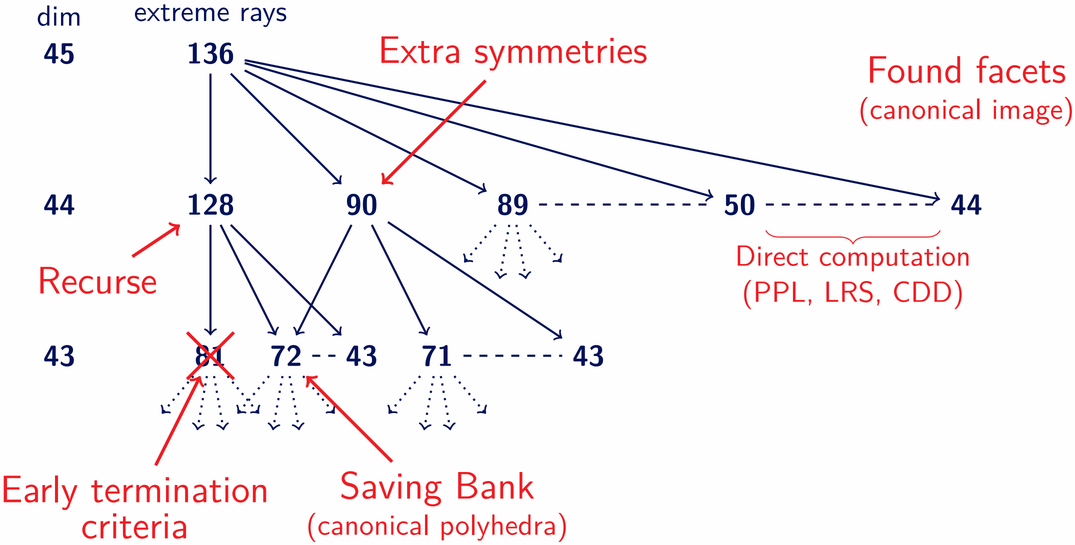


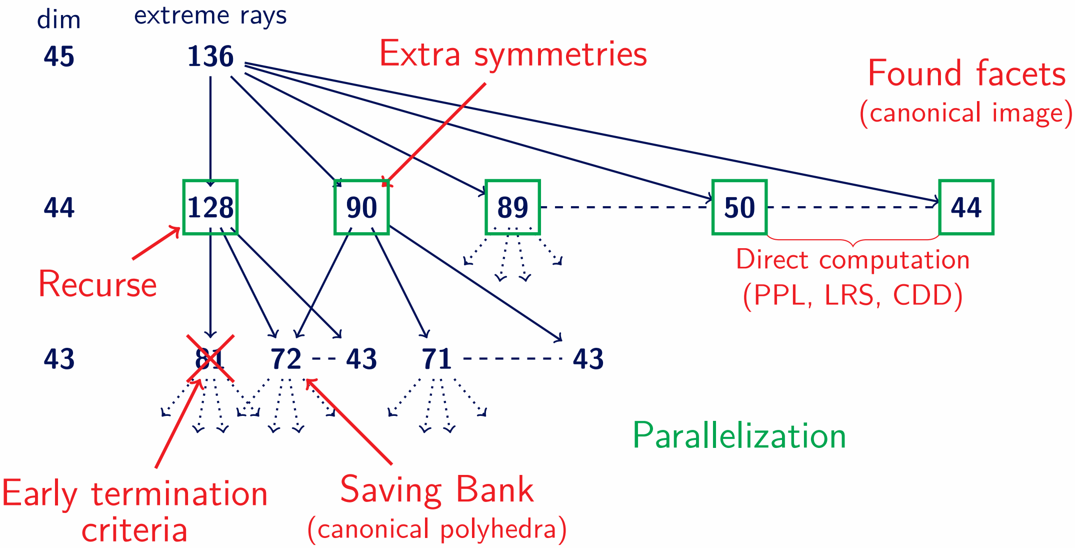
Recursive Adjacency Decomposition Method



Recursive Adjacency Decomposition Method



Recursive Adjacency Decomposition Method



Results

Lattice packing problem in dimension 9

$\pm$ 3 000 000 core hours later...

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Theorem: Main result

There are precisely 2 237 251 040 non-similar perfect forms in dimension **9**.

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The Laminated lattice $\mathbf{\Lambda_9}$ is the unique densest lattice packing in dimension $\mathbf{9}$.

The Hermite constant in dimension $\mathbf{9}$ is $\gamma_{\mathbf{9}} = \mathbf{2}$.

All perfect forms by their kissing number

$\lvert\min(Q)\rvert/2$	#	$\lvert\min(Q)\rvert/2$	#	$\lvert\min(Q)\rvert/2$	#
45	1 353 947 672	**61**	2 244	**77**	1
46	471 756 975	**62**	1 713	**78**	1
47	267 588 732	**63**	641	**79**	2
48	84 473 357	**64**	634	**80**	12
49	37 278 163	**65**	236	**81**	3
50	13 324 560	**66**	203	**82**	4
51	5 299 974	**67**	172	**84**	2
52	2 009 292	**68**	74	**85**	2
53	903 943	**69**	44	**88**	1
54	366 796	**70**	42	**90**	2
55	155 182	**71**	26	**91**	1
56	78 919	**72**	21	**99**	1
57	31 113	**73**	7	**129**	1
58	17 207	**74**	3	**136**	1
59	8 231	**75**	4		
60	4 820	**76**	6		

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99.9991% of all forms

$<$ **5%** of runtime.

High incidence cases

For comparison: $\boldsymbol{\mathcal{P}(Q_{E_8})}$ now takes $\mathbf{9}$ core hours (before $\pm\mathbf{1}$ month).

High incidence cases

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Table: Cost of dual description cases with more than **50**k core hours. These cases account for **1.5** million of the total amount of **2** million core hours spent on dual description instances.

$\lvert\min(Q)\rvert/2$	Core hours	$\lvert\mathrm{linaut}(P)\rvert$	rays (orbits)	$\lvert\mathrm{aut}(Q)\rvert$	neighbours (orbits)
136	59 277	660 602 880	64 001 686	10 321 920	1 038 153 863
84	75 467	12 288	171 496 157	384	1 514 557 045
99	84 197	589 824	137 739 671	18 432	1 842 205 495
90	85 349	73 728	185 824 962	2 304	2 058 568 310
74	95 784	128	333 146 387	16	1 257 559 244
80	97 118	7 680	108 828 919	480	764 775 430
81	181 570	1 296	254 734 260	2 592	254 734 260
80	219 437	128	772 745 513	256	772 745 513
82	245 030	432	680 747 757	864	680 747 757
76	355 554	24	1 549 616 491	48	1 549 616 491

Hard to reach perfect forms

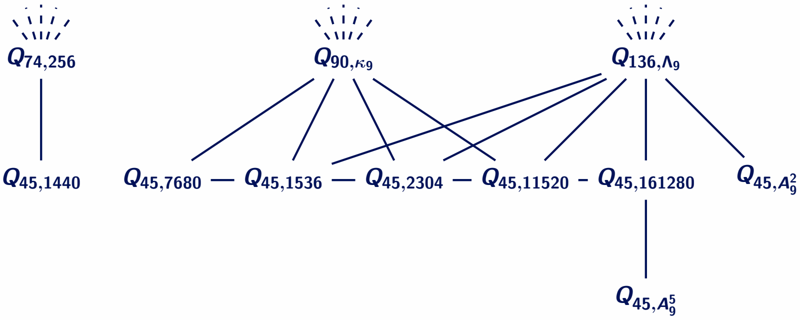

Figure: Part of Voronoi graph showing all perfect forms that are only connected via high-incidence perfect forms.

- All other forms are connected via forms with $|\mathrm{Min}\, \boldsymbol{Q}| \leq \mathbf{2 \cdot 58}$.

Thank you!

Hopefully soon on arxiv...

Thank you!