Cryptanalysis of rank-2 module-LIP: a single real embedding is all it takes

Bill Allombert, Alice Pellet-Mary (Université de Bordeaux),
Wessel van Woerden (Université de Bordeaux & PQShield) .

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This talk: What about other number fields?

Totally real | What about here? | Totally imaginary

HAWK (CM field)

Number fields and module-LIP

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- Cause of most technical problems in this work

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[MPMPW24]

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- This is a lattice problem!

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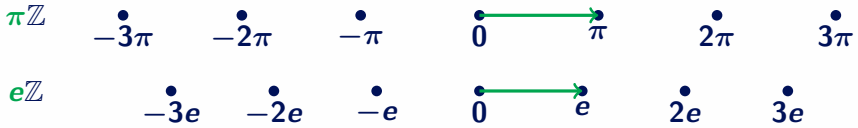

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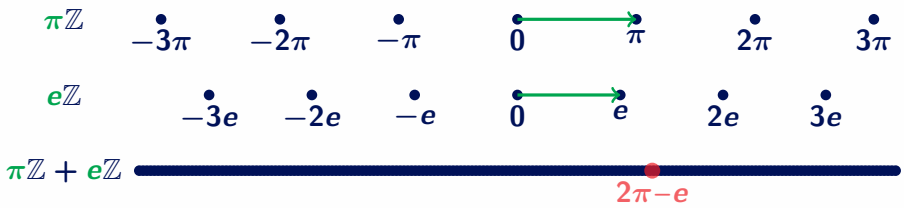

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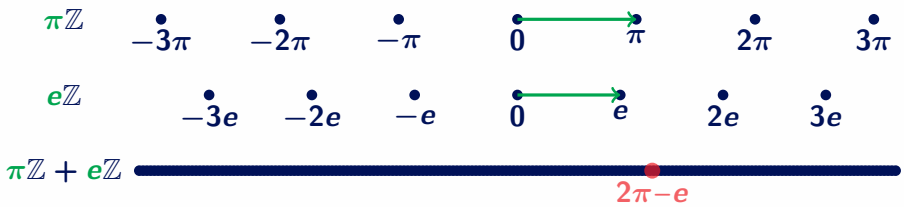

Small combinations only ($S = \{-2, -1, 0, 1, 2\}$)

Recovery from a single real embedding

integer combination π and e close to v

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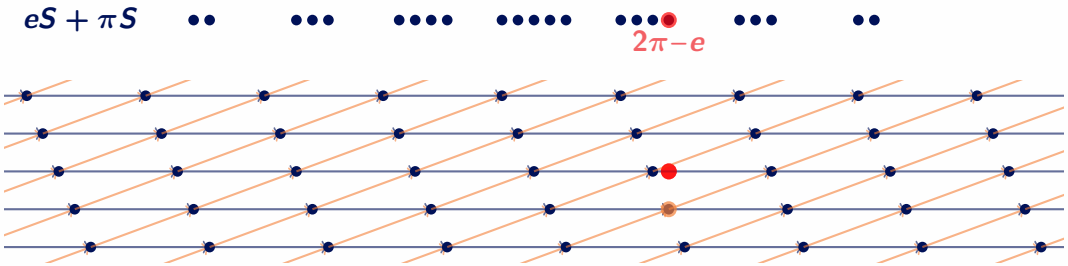

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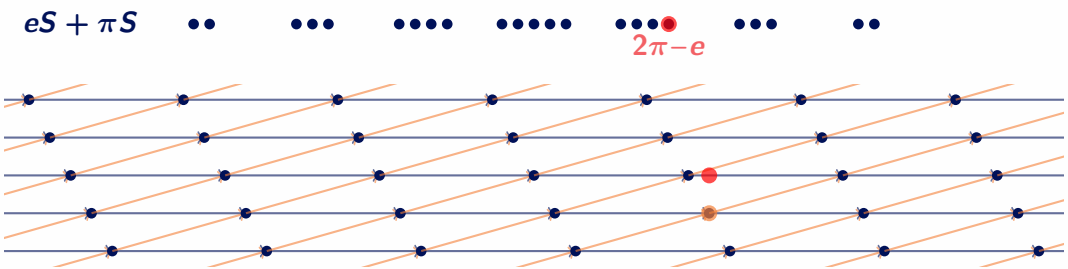

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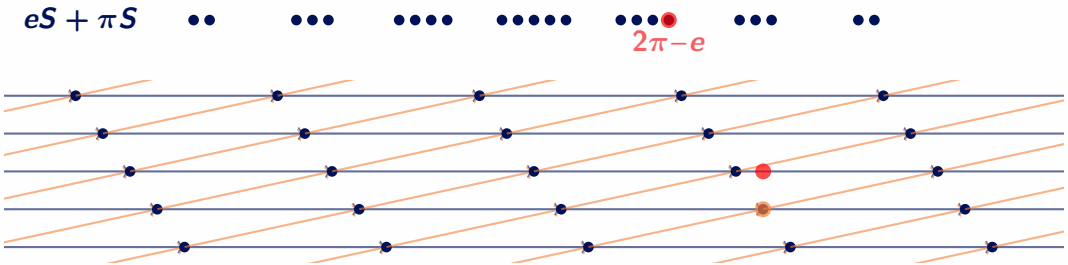

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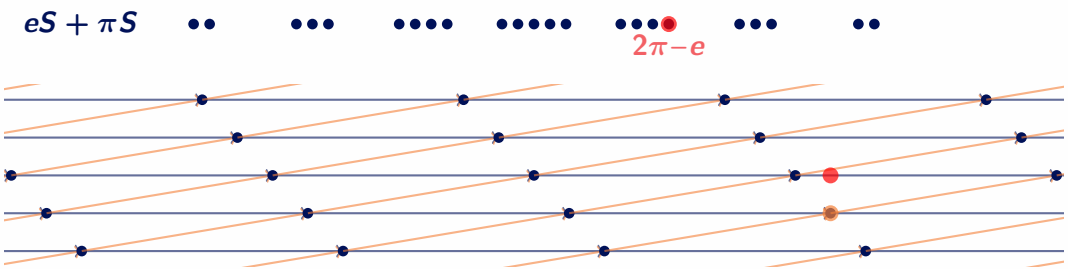

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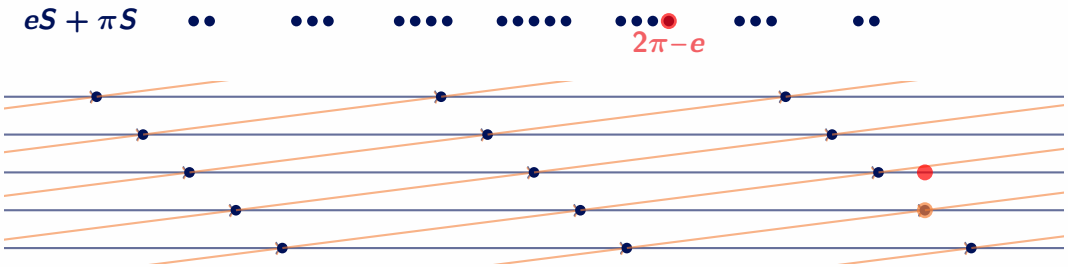

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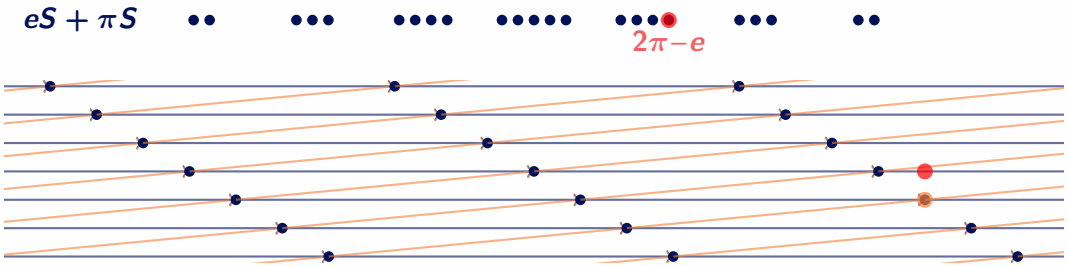

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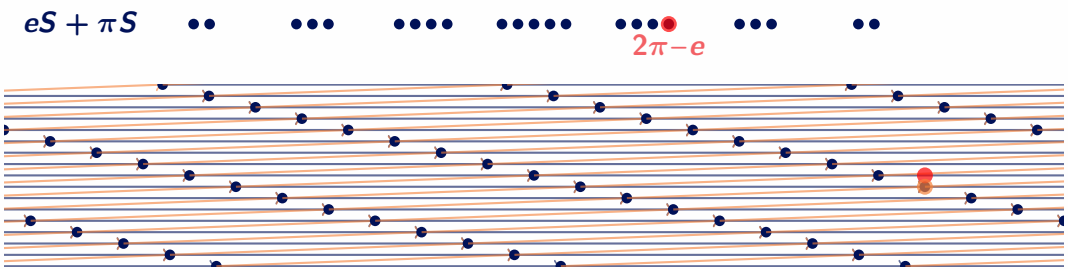

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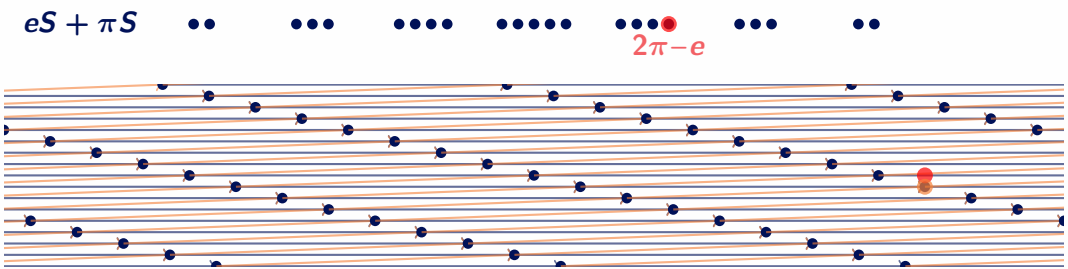

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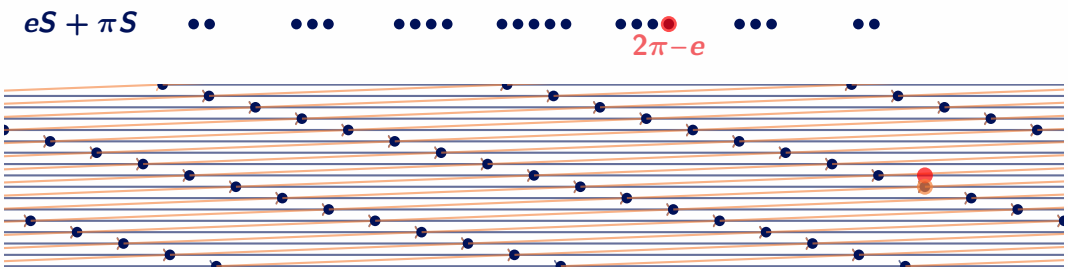

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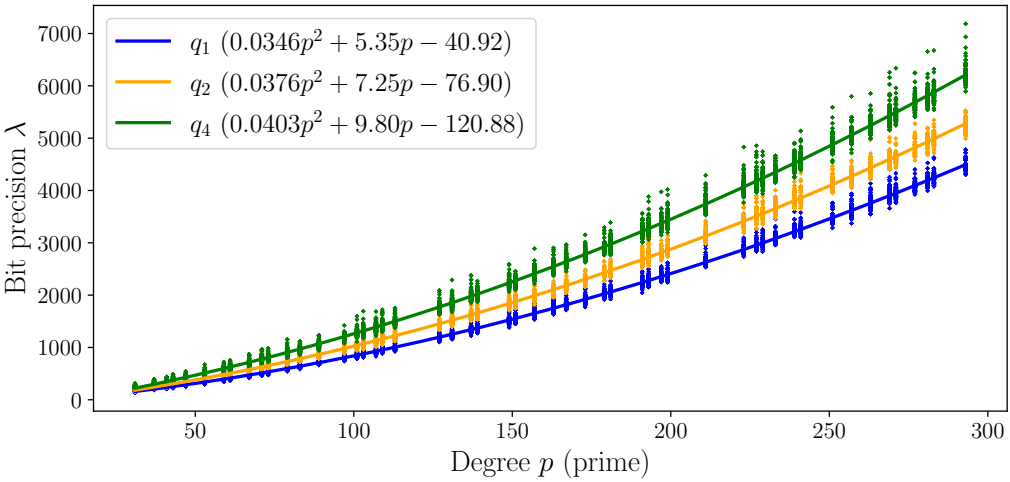
Required precision for NTRU Prime field

Lemma [PMS21]: $\lambda \geq \mathbf{poly}(\log(|\Delta_K|), \log \|P\|, \log(\|x\|))$ is sufficient



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- Use trick from previous talk to recover ideal $(a+bi)\mathcal{O}_L$

Notation: $z = a + bi$, $z' = c + di$

Lemma: Ideal recovery (previous talk!)

$$z\mathcal{O}_L = \mathcal{O}_L \cap zz'^{-1}\mathcal{O}_L = \mathcal{O}_L \cap q_1(\det(B)i + q_2)^{-1}\mathcal{O}_L.$$

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When L is a CM field Gentry-Szydlo recovers z from $z\mathcal{O}_L$ and $\overline{z}z$.

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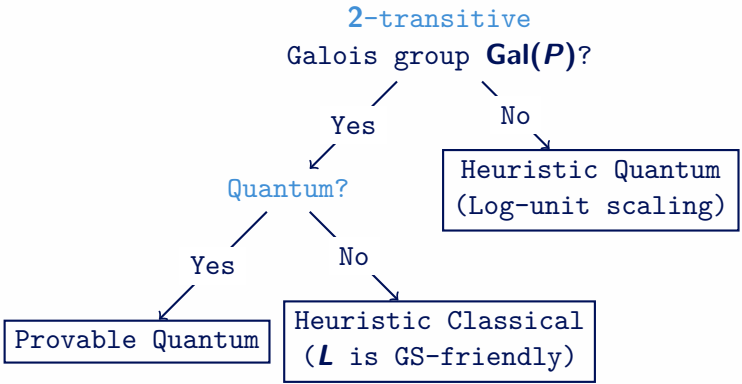
We have 3 approaches to solve this.

Make your choice



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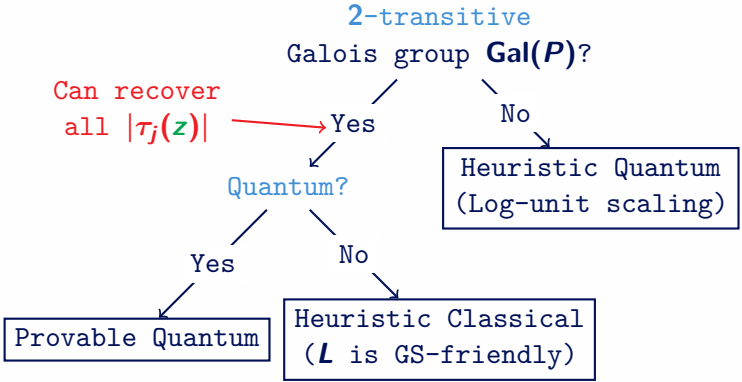

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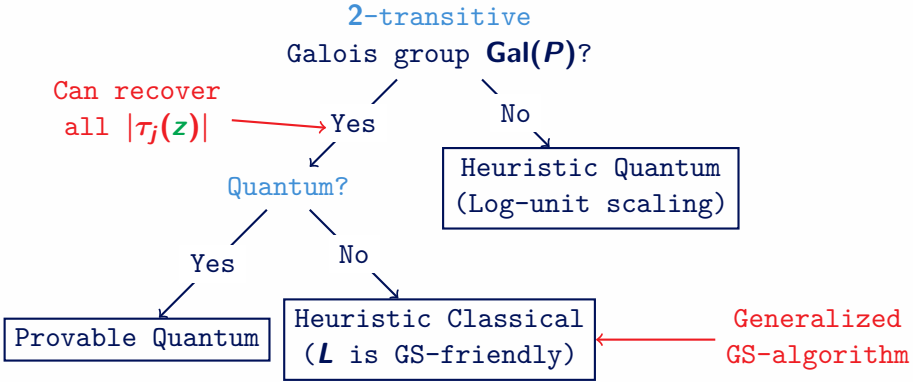

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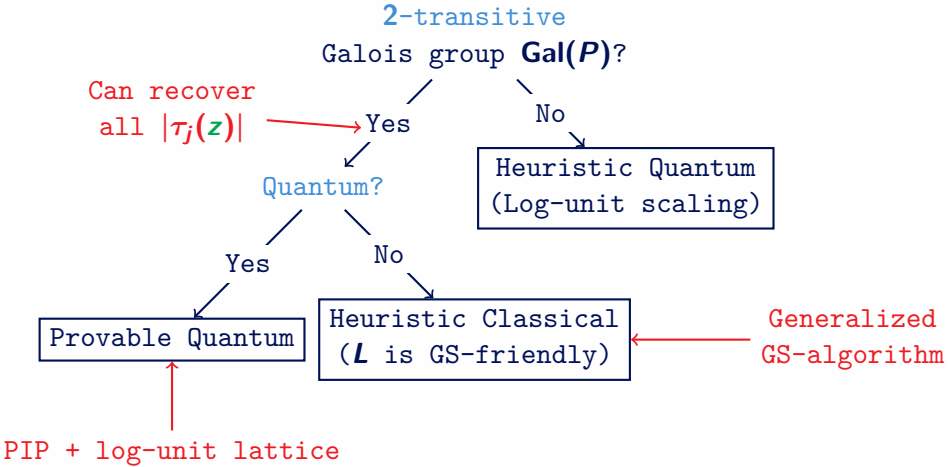

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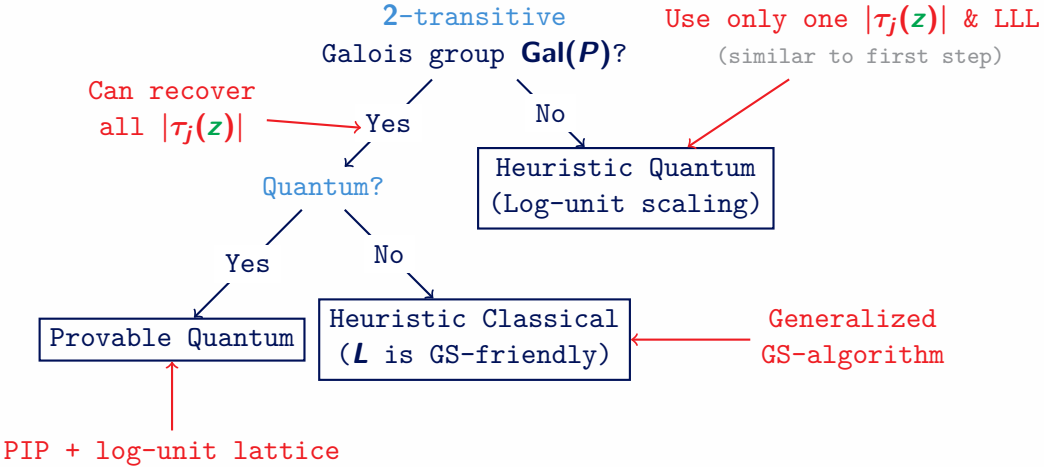

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