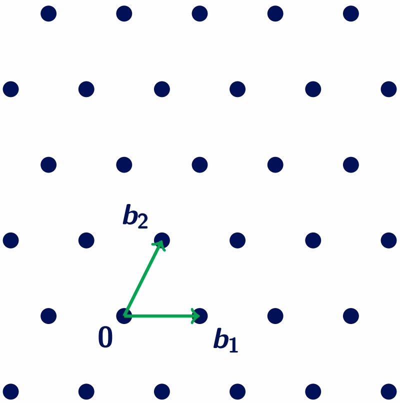
On the existence of good lattice packings and smoothing within a fixed genus

Wessel van Woerden (Université de Bordeaux, IMB, Inria).

université de BORDEAUX

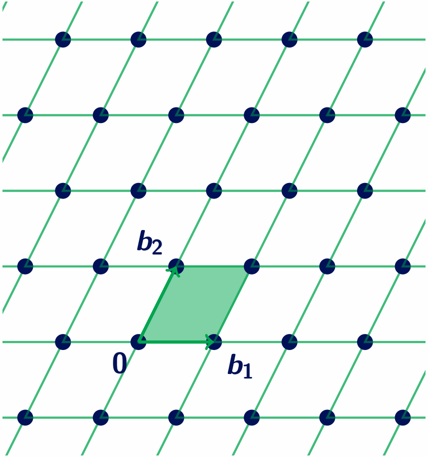


Lattice

$\mathbb{R}$-linearly independent $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$

$$\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n,$$

basis B, gram matrix $G := B^\top B$



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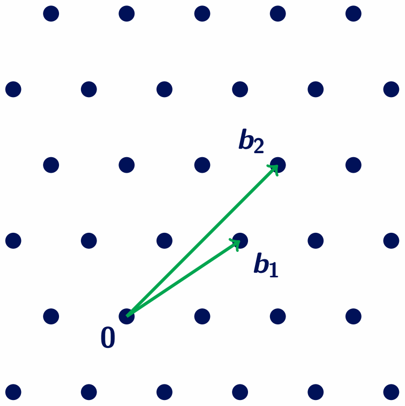
$$\det(\mathcal{L}) := \mathrm{vol}(\mathbb{R}^n/\mathcal{L}) = |\det(B)|$$



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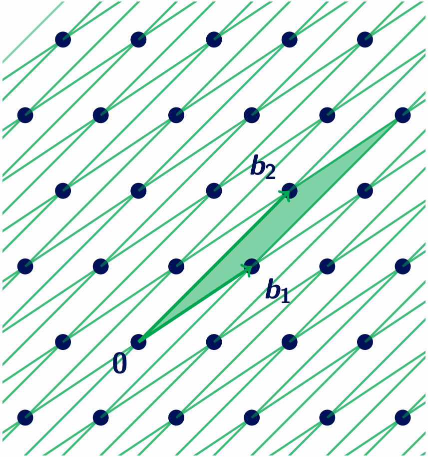

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$q \geq 2, 0 < n < m$, let $A \leftarrow \mathcal{U}(\mathbb{Z}^{m\times n})$ and consider $\mathcal{L} = A\mathbb{Z}^n + q\mathbb{Z}^m \subset \mathbb{Z}^m$.

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The Haar measure on $\mathcal{SL}_n(\mathbb{R})$ has finite mass on the quotient space of unit volume lattices $\mathcal{SL}_n(\mathbb{R})/\mathcal{SL}_n(\mathbb{Z})$.

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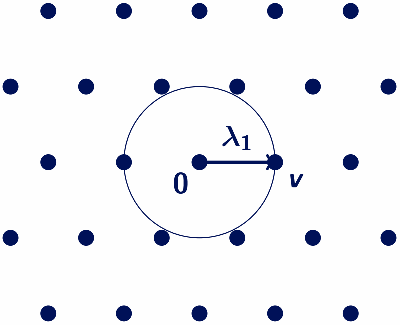
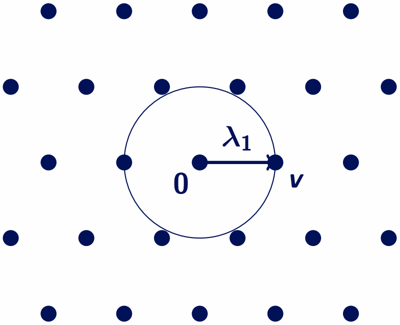
- Mathematically elegant and useful for certain proofs



First minimum & theta series

$$\lambda_1(\mathcal{L}) := \min_{x \in \mathcal{L} \setminus \{0\}} \|x\|_2$$

$$\theta_{\mathcal{L}}(q) := \sum_{x \in \mathcal{L}} q^{\|x\|^2} = 1 + N_{\lambda_1} q^{\lambda_1^2} + \dots$$



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$$\lambda_1(\mathcal{L}) \leq \underbrace{2 \frac{\det(\mathcal{L})^{1/n}}{\mathtt{vol}(\mathcal{B}^n)^{1/n}}}_{\mathsf{Mk}(\mathcal{L})} \approx \sqrt{2n/\pi e}\, \det(\mathcal{L})^{1/n}$$

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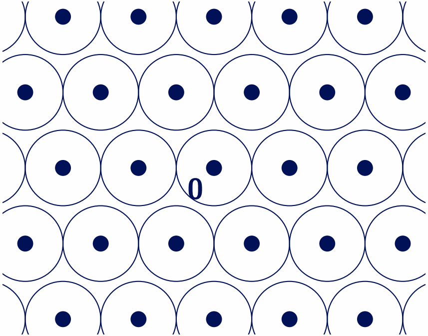

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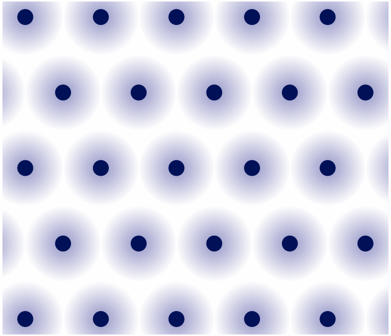
Minkowski-Hlawka Theorem:

For random lattices $\mathbb{E}[\lambda_1(\mathcal{L})] = \mathsf{gh}(\mathcal{L}) := \frac{1}{2}\mathsf{Mk}(\mathcal{L}) \approx \sqrt{n/2\pi e}\cdot\det(\mathcal{L})^{1/n}$.

$\Rightarrow$ there exists a lattice with $\lambda_1(\mathcal{L}) \ge \mathsf{gh}(\mathcal{L})$ ($\exists$ good lattice packing)



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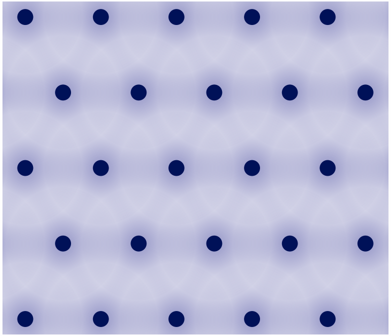

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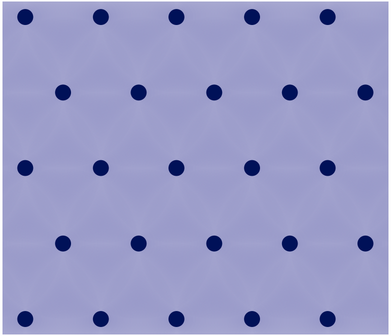

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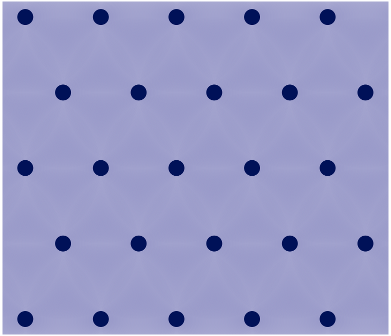

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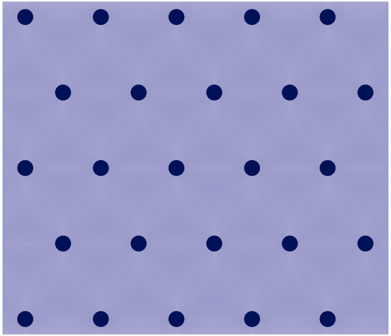
$$\eta_\varepsilon(\mathcal{L}) = \min\{s > 0 : \theta_{\mathcal{L}^*}(\exp(-\pi s^2)) \leq 1 + \varepsilon\}$$

Dual lattice

$$\mathcal{L}^* := \{y \in \mathbb{R}^n : \forall x \in \mathcal{L}, \langle x, y \rangle \in \mathbb{Z}\}$$

$$\eta_{2^{-n}}(\mathcal{L}) \leq \sqrt{n}/\lambda_1(\mathcal{L}^*)$$



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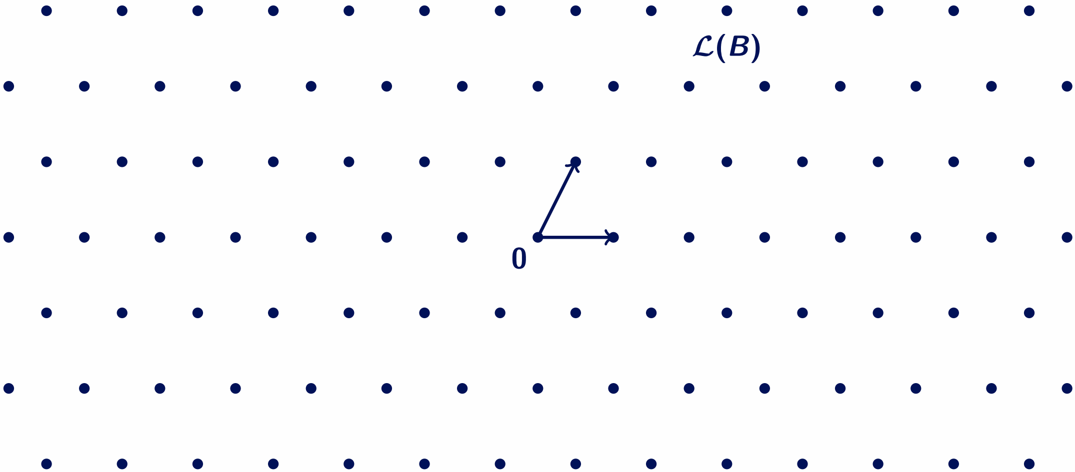
Good smoothing: $\epsilon \in (e^{-n}, 1]$

For a random lattice $\mathcal{L}^*$, $\theta_{\mathcal{L}^*}(\exp(-\pi s^2)) \leq 1 + O(ns^{-n}\det(\mathcal{L}))$

$\Rightarrow$ there exists a lattice with $\eta_\varepsilon(\mathcal{L}) \leq \theta(n\det(\mathcal{L})/\varepsilon)^{1/n}$.

Lattice Isomorphism Problem (LIP)



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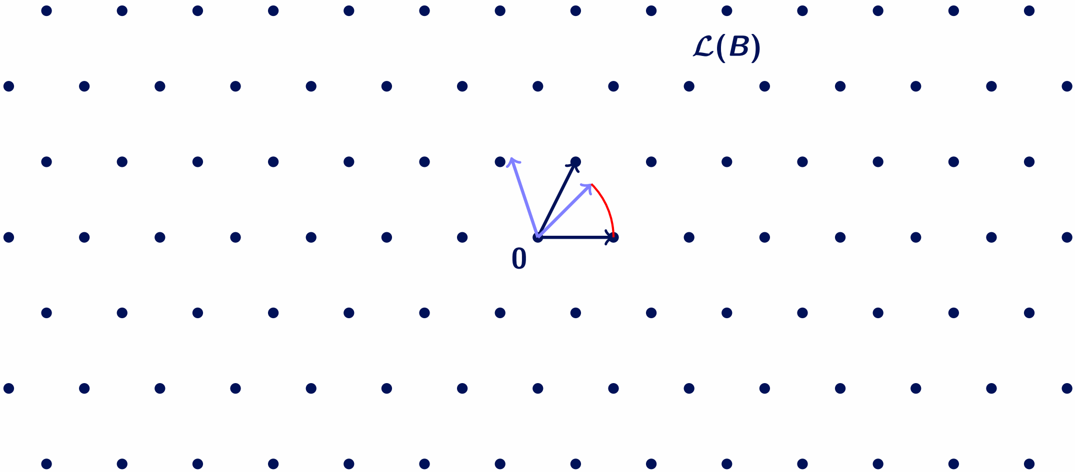

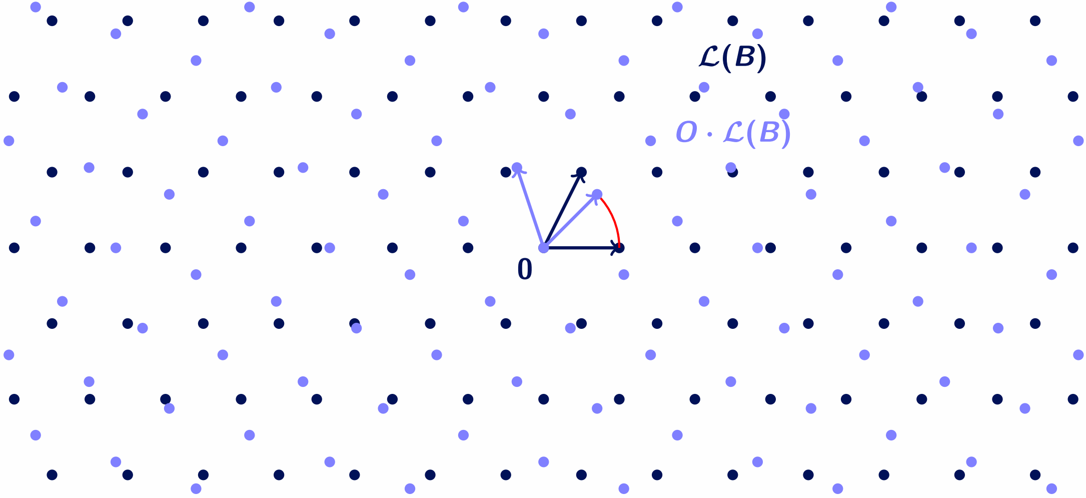
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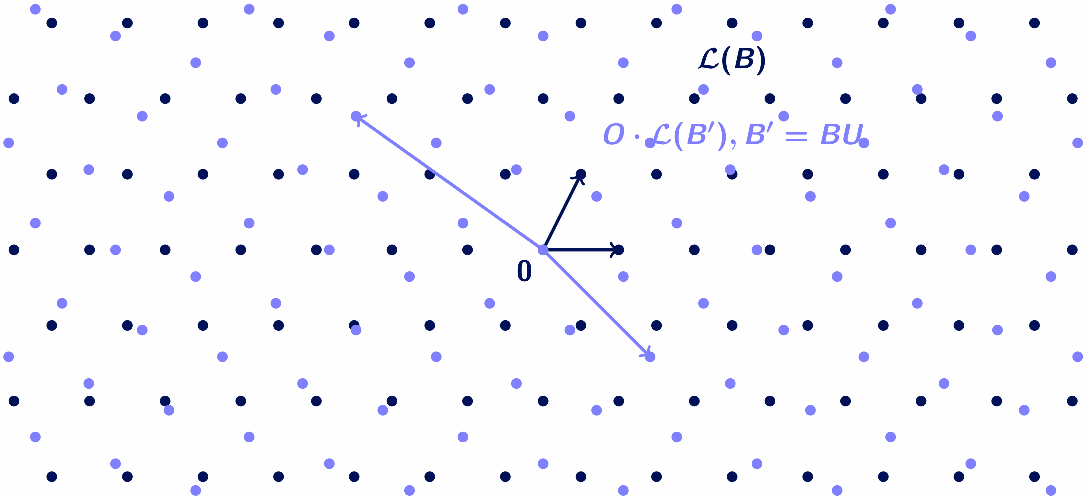

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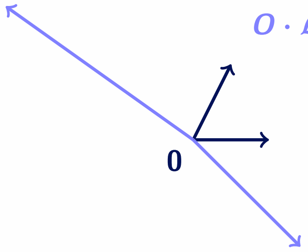

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- Restrict to integral or rational gram matrices

Cryptography from LIP

- LIP as a new hardness assumption

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Ducas & vW: On LIP, QFs, Remarkable Lattices, and Cryptography

Use LIP to hide a remarkable lattice:

- Identification, Encryption and Signature scheme

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Ducas et al.: HAWK scheme

Efficient signature scheme based on module-LIP on $\mathbb{Z}^n$

- submitted to NIST call for additional signatures

- Several others works using LIP appeared recently

Distinghuish LIP

Definition: distinguish LIP ($\boldsymbol{\Delta}$-LIP)

Let $\mathcal{L}_1, \mathcal{L}_2$ be two non-isomorphic lattices and let $\boldsymbol{b} \leftarrow \{1, 2\}$ uniform.
Given $\mathcal{L} \in [\mathcal{L}_{\boldsymbol{b}}]$, recover $\boldsymbol{b}$.

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Usual security assumption:

Given:

1. some remarkable lattice $\mathcal{L}_1$
2. an auxiliary lattice $\mathcal{L}_2$ with certain (good) geometric properties

Then: cryptographic scheme is secure if Δ-LIP on $\mathcal{L}_1, \mathcal{L}_2$ is hard.

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Goal: find an auxiliary lattice with the right properties

Arithmetic Invariants ($\mathbf{ari}(\mathcal{L})$)

- $\mathbf{det}(\mathcal{L}) = \mathbf{det}(\mathcal{L}_b)$.
- $\mathbf{gcd}(\mathcal{L}) := \mathbf{gcd}\{\langle x, y\rangle : x, y \in \mathcal{L}\}$
- parity $\mathbf{par}(\mathcal{L}) = \mathbf{gcd}\{\|x\|^2 : x \in \mathcal{L}\} / \mathbf{gcd}(\mathcal{L})$

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- Equivalence over $R \supset \mathbb{Z}$, $U \in \mathrm{GL}_n(R)$, $R \in \{\mathbb{R}, \mathbb{Q}, \forall p\ \mathbb{Q}_p, \underbrace{\forall p\ \mathbb{Z}_p}_{\text{Genus}}\}$

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$\Rightarrow$ auxiliary lattice must have same invariants

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  $\Leftrightarrow U^\top G_1 U = G_2$ for $U \in \mathcal{GL}_n(\mathbb{Z}_p)$.
- Covers all the other known arithmetic invariants

Motivation

Ducas & vW: On LIP, QFs, Remarkable Lattices, and Cryptography

Instantiation blows up geometric gaps from f to $O(f^2)$ or $O(f^3)$. If there exists a lattice $\mathcal{L}_2 \in \mathbf{gen}(\mathcal{L}_1)$ with geometric gaps of $O(1)$ then this reduces to $O(f)$. $(\mathrm{gh}(\mathcal{L})/\lambda_1(\mathcal{L}) = O(1)$ for $\mathcal{L} = \mathcal{L}_2, \mathcal{L}_2^*)$

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Bennett et al.: Just how hard are rotations of $\mathbb{Z}^n$?

Do there exist lattices in $\mathsf{gen}(\mathbb{Z}^n)$ with $\lambda_1(\mathcal{L}) \geq \Omega(\sqrt{n/\log(n)})$ or with $\eta_\varepsilon(\mathcal{L}) \leq \eta_\varepsilon(\mathbb{Z}^n)/\sqrt{\log(n)} \approx \sqrt{\log(1/\varepsilon)/\log(n)}$ for $\varepsilon < n^{-\omega(1)}$?

Ducas & vW: On LIP, QFs, Remarkable Lattices, and Cryptography

Instantiation blows up geometric gaps from f to $O(f^2)$ or $O(f^3)$. If there exists a lattice $\mathcal{L}_2 \in \mathrm{gen}(\mathcal{L}_1)$ with geometric gaps of $O(1)$ then this reduces to $O(f)$. ($\mathrm{gh}(\mathcal{L})/\lambda_1(\mathcal{L}) = O(1)$ for $\mathcal{L} = \mathcal{L}_2, \mathcal{L}_2^*$)

Bennett et al.: Just how hard are rotations of $\mathbb{Z}^n$?

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Ackermann, Wallet, et al.: to appear

Conjecture: for $n \geq 85$ there exists at least one $\mathcal{L} \in \mathrm{gen}(\mathbb{Z}^n)$ such that $\lambda_1(\mathcal{L}) \geq \sqrt[4]{72n}$. (needed to instantiate their PKE security proof)

Results (1) - Good (dual) packing

Theorem (good packing): Minkowski-Hlawka theorem for fixed genus

Let $\mathcal{G}$ be any genus of dimension $n \geq 6$ such that $p^{n-5} \nmid \det(\mathcal{G})^2$ for all primes p. Then there exists a lattice $\mathcal{L}^* \in \mathcal{G}$ with $\lambda_1(\mathcal{L})^2 \geq \lceil \Theta(\omega_n / \det(\mathcal{L}))^{-2/n} \rfloor \approx n/2\pi e \cdot \det(\mathcal{L})^{2/n} = \mathrm{gh}(\mathcal{L})^2$.

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- Essentially matches packing density of a random lattice
- Similar result for simultaneous good primal and dual packing.
- Requirement that $p^{n-5} \nmid \det(\mathcal{G})^2$ can be replaced by a milder but more technical condition. (or removed at a small loss)

Results (2) - Good smoothing

Theorem (good smoothing):

'...' Let $\varepsilon \in [e^{-n}, 1)$, then there exists a lattice $\mathcal{L} \in \mathcal{G}$ with $\eta_\varepsilon(\mathcal{L}^*) \leq \theta(n \det(\mathcal{L}^*)/\varepsilon)^{\frac{1}{n}}$.

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- Same conditions as previous result
- Essentially matches smoothing of random lattice
- Works even for constant ε
- Also works for $\varepsilon < e^{-n}$ but smoothing for such cases is essentially determined by $\lambda_1(\mathcal{L}^*)$.

The tool: mass formulas (1)

Definition: distribution over Genus

Consider the mass function w given by $w(\mathcal{L}) = 1/|\mathrm{Aut}(\mathcal{L})|$. For a genus $\mathcal{G}$ let $\mathcal{D}(\mathcal{G})$ be the distribution where each isomorphism class $[\mathcal{L}]$ is sampled with relative weight $w(\mathcal{L})$.

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- Lemma: $|\mathcal{G}| \geq 2M(\mathcal{G})$.

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- Surprisingly the average is efficient to compute
- Old but not well known result (by experimental validation)

Example: even unimodular case (1)

Definition: even unimodular lattices

The genus $\mathcal{G}_{n,e}$ of n-dimensional even unimodular lattices consists of all integral lattices of determinant 1 and even parity.

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Lemma: mass formula

For $n = 8k \geq 8$ we have

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where B_i is the i-th Bernoulli number, and $\sigma_z(m) = \sum_{d|m} d^z$ is the sum of positive divisors function.

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- $\Theta_{\mathcal{G}_{128,e}}(q) = 1 + 6.11 \cdot 10^{-37} q^2 + 5.64 \cdot 10^{-18} q^4 + 7.00 \cdot 10^{-7} q^6 + 52.01 q^8 + 6.63 \cdot 10^7 q^{10} + O(q^{12})$

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Lemma: existence of good packing

Let $\mathcal{G}$ be a genus with average theta series $\Theta_{\mathcal{G}}(q) = 1 + \sum_{m=1}^{\infty} N_m q^m$.
If $\sum_{m=1}^{\lambda} N_m < 2$, then there exists a lattice $\mathcal{L} \in \mathcal{G}$ s.t. $\lambda_1(\mathcal{L})^2 > \lambda$.

Lemma: even packing (Serre)

Let $n = 8k \geq 8$ with $k \in \mathbb{N}$, then there exists an n-dimensional even unimodular lattice $\mathcal{L}$ with $\lambda_1(\mathcal{L})^2 \geq 2 \cdot \left\lceil \frac{1}{2}(\frac{3}{5}\omega_n)^{-2/n} \right\rceil \approx n/2\pi e$.

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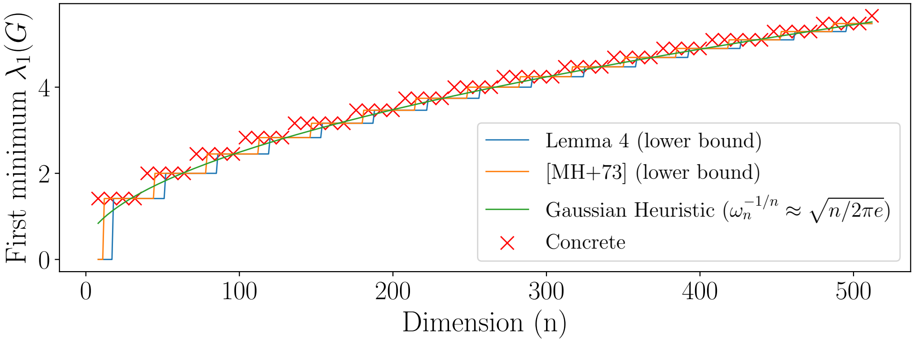

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- More generally: sum over primal and dual theta series to lower bound $\lambda_1(\mathcal{L})$ and $\lambda_1(\mathcal{L}^*)$ simultaneously.
- Not aware of similar results for other genera

- Recall: for the smoothing parameter $\eta_\varepsilon(\mathcal{L}^*)$ we need to bound $\theta_{\mathcal{L}}(\exp(-\pi s^2)) \leq 1 + \varepsilon$.
- Note that for any $s > 0$ we have

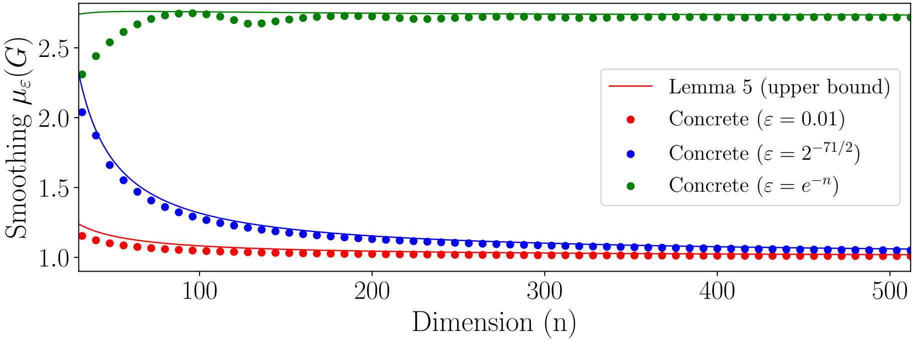
$$\theta_{\mathcal{G}}(\exp(-\pi s^2)) = \mathbb{E}_{[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})} \left[\theta_{\mathcal{L}}(\exp(-\pi s^2))\right]$$

Lemma: existence of good smoothing

For any genus $\mathcal{G}$, let $\varepsilon > 0$ and let $s > 0$ be such that $\Theta_{\mathcal{G}}(\exp(-\pi s^2)) \leq 1 + \varepsilon$, then there exists a lattice $\mathcal{L} \in \mathcal{G}$ such that $\eta_\varepsilon(\mathcal{L}^*) \leq s$.

Lemma: even smoothing

Let $n = 8k \geq 8$ with $k \in \mathbb{N}$, and $\varepsilon \in [e^{-n}, 1)$, then there exists an n-dimensional even unimodular lattice $\mathcal{L}$ with $\eta_\varepsilon(\mathcal{L}) \leq (\pi\varepsilon/n)^{-\frac{1}{n+2}}$.



General case: compute mass formula

- Note that we want to count the average number of solutions N_y to $f(x) := x^\top G_{\mathcal{L}} x = y$ with $x \in \mathbb{Z}^n$ over the randomness of $[\mathcal{L}] \leftarrow \mathcal{D}(\mathcal{G})$.

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Theorem: Siegel-Weil mass formula

For any genus $\mathcal{G}$ of dimension ≥ 2 and average theta series $\Theta_{\mathcal{G}}(q) = 1 + \sum_{y=1}^{\infty} N_y q^m$ we have

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- Local-global principle
- Only primes $p | 2y \det(\mathcal{G})^2$ have to be considered
- Can even be generalized to matrix equations!

  (mass formula from $M(\mathcal{G})$ follows from equation $U^\top G U = G$)

Local density over reals

- For $f(x) = x^\top x = x_1^2 + \ldots + x_n^2 \in \mathbb{R}$ the density of solutions for $f(x) = y$ is essentially the volume of the sphere of radius $\sqrt{y}$.

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- Behaves as expected:

  sparser lattice $\Rightarrow$ larger $\det(\mathcal{G})$ $\Rightarrow$ smaller coefficients.

Local density over $\mathbb{Z}_p$

- This is where things get complicated, count solutions mod p^k

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Lemma: Local densities at $\mathbb{Z}_p$ are bounded

Let $\mathcal{G}$ be a genus with $p^{n-5} \nmid \det(\mathcal{G})^2$ for all primes p, then for all $y \geq 0$ we have

$$\prod_{p=2,3,\ldots} \delta_{\mathcal{G},p}(y) \leq \frac{18\zeta(2)}{7\zeta(3)} < 3.52$$

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Rough idea:

$G \sim_{\mathbb{Z}_p} G_0 + p \cdot G_1 + p^2 \cdot G_2 + \ldots$ with $\det(G_i) \neq 0 \bmod p$, $\dim(G_0) \geq 6$
If $\delta_{G_0,p}(y) \leq c$ for all $y \geq 0$, then $\delta_{G,p}(y) \leq c$ for all $y \geq 0$.

Theorem:

Let $\mathcal{G}$ be any genus with $p^{n-5} \nmid \det(\mathcal{G})^2$ for all primes p. Let $\Theta_{\mathcal{G}}(q) = 1 + \sum_{y=1}^{\infty} N_y q^y$ be its average theta series, then for $y \geq 1$ we have

$$N_y \leq 3.52 \cdot \delta_{\mathcal{G},\infty}(y) \leq 1.76 \cdot n\omega_n y^{n/2-1} \cdot \det(\mathcal{G})^{-1}$$

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- Conjecture: remove conditions $\implies$ extra factor $\mathrm{poly}(y)$
  (but rather tedious to work out)

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Open questions:

- What about other geometric properties?
- What else can we do with these mass formulas?

Thank you! :)

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